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COMPUTER METHODS

Some Modifications of Integer Optimization Problems with Uncertainty and Risk

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Abstract—This paper describes various approaches to formalizing a certain class of limited resource management problems, developing numerical algorithms for selecting an optimal integer solution, and evaluating its efficiency in using it in relation with the stock market. We propose integer models and methods for assessing these models under the deterministic or interval-type future price of assets. Also, we present stability analysis methods for the optimal solution. The optimal choice solutions based on the classical portfolio theory and the author's concept are compared. Based on the comparison, it is concluded that the approach and numerical method proposed below are correct and are more efficient to apply to these optimization problems than the traditional methods and algorithms.

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INTRODUCTION

This paper considers integer single-criterion and two-criteria optimization problems of managing limited financial resources under uncertainty and risk and their solution methods. Due to the enumerative nature of these problems, we propose branch-and-bound methods based on our original algorithms of calculating the upper, lower, and current upper estimates for their solution. Also, we propose stability analysis methods for the optimal solutions under the initial data variations. The mathematical framework for the optimal choice in the stock market developed below extends the classical Markowitz–Sharpe theory; for example, see [1]. The direct practical application of these models is related to choosing the optimal production program of an enterprise, determining the purchase structure of material resources of an industrial enterprise, and calculating the optimal portfolio of indivisible financial assets. The classical optimization problem in the stock market consists of choosing assets with a total value not exceeding the investor's budget to maximize the return under limited risk or, conversely, minimize risk under a limited expected return. For a financial asset, this concerns the average return over the observed time interval, and the risk is understood as the standard deviation of the expected return from the average value. This measure of risk is based on the law of large numbers and Chebyshev's inequality: the smaller the standard deviation of the return on a financial asset the lower the probability of its deviation from the average value.

Initially, the theory of the optimal choice (e.g., see [1–5]) was developed for portfolios of financial assets. Later on, it has been increasingly used to study and assess the efficiency of projects portfolios in the real sector of the economy, organize wholesale purchases of heterogeneous goods for retail networks, purchase material resources of production, etc. In particular, these approaches were described in [1, 6–8].

Stochastic methods and statistical data allow estimating the return and risk on a portfolio. Classical portfolio choice models (the Markowitz model and the capital asset pricing model (CAPM)) proceed from the assumption that the financial assets in a portfolio are infinitely divisible. Therefore, the portfolio formation problem was settled by obtaining the shares of assets purchased in the optimal solution.

This approach is applicable if the price of a stock (or a lot of homogeneous securities) is smaller than the investment budget. If this condition is not met, the resulting solution may not only be suboptimal but even inadmissible. Attempts of analysts to obtain the optimal solution by rounding the solution components far from always give the desired result. Due to this, portfolio efficiency is often analyzed using not only continuous (classical) models but also their integer analogs [9, 10].

This problem was considered by several researchers. We mention the publications [7, 9, 11] and the work [4], where the discrete optimal investment portfolio problem was reduced to the *NP*-complete Turing problem. Studies in this area were topical in the 1950s, the era of low-performance electronic computers, and were accompanied by strong interest in constructive algorithms for solving *NP*-complete problems. It is topical to solve this class of problems in real time and make operational adjustments not only in the information database but sometimes in the algorithms and problem statements.

Note that in the practical optimal choice, *NP*-complete problems, such as the nonlinear discrete high-dimensional problems, can be effectively solved considering the peculiarities of their statements, criteria, and constraints; for example, see the discussion in [12, 13]. The issues of determining exact and approximate solutions of *NP*-hard problems were considered in [14–16]. This paper continues those investigations. We develop and verify integer variants of optimal choice models and develop a framework for determining the stability domain of optimal solutions of integer problems.

1. A MATHEMATICAL OPTIMIZATION MODEL WITHOUT RISK

The problem statement is as follows. Let an investor possess money in volume F on the time interval $[0, T]$. He can purchase n types of securities in lots. Each lot contains securities (stocks) of one type. The quantity of securities in lot i, $i=\overline{1,n}$, is V_i. At the time instant $t=0$, the initial price of one security of type i is α_i, and its future price at the time instant $t=T$ is defined stochastically, equaling γ_i^j with the probability P_j, $j=\overline{1,k}$. It is required to purchase lots of securities to maximize the profit obtained by selling them at the time instant T. Formally, this problem can be written as

$$\sum_{i=1}^{n} V_i x_i \overline{\gamma}_i + \left(F - \sum_{i=1}^{n} V_i x_i \alpha_i\right) \to \max, \tag{1.1}$$

$$\sum_{i=1}^{n} V_i x_i \alpha_i \le F, \tag{1.2}$$

$$\overline{\gamma}_i = \sum_{j=1}^{k} \gamma_i^j P_j, \tag{1.3}$$

$$\sum_{j=1}^{k} P_j = 1,$$

$$P_j \ge 0, \tag{1.4}$$

$$x_i \in \{0;1\}, \quad i=\overline{1,n}. \tag{1.5}$$

If lot i is purchased, then $x_i = 1$; otherwise, $x_i = 0$. Problem (1.1)–(1.5) is a generalization of the well-known knapsack problem [8].

The objective function in this problem consists of two components. The first component is the sales proceeds of securities at the price γ_i, and the second one is the balance of the money after forming the securities portfolio. Since F does not affect the optimal solution, we obtain the objective function

$$\sum_{i=1}^{n} V_i x_i (\overline{\gamma}_i - \alpha_i) \to \max. \tag{1.6}$$

Problem (1.1)–(1.5) belongs to the class of discrete optimization problems with Boolean variables and is *NP*-hard [8]. It can be solved using the following branch-and-bound scheme with our original algorithm for calculating the upper, lower, and current upper estimates:

1. Calculation of the upper estimate. The value $\overline{\gamma}_i/a_i$ is calculated for all lots of stocks. Let us renumber all lots as follows: $\overline{\gamma}_1/a_1 \ge \overline{\gamma}_2/a_2 \ge \ldots \ge \overline{\gamma}_n/a_n$. Later financial resources will be allocated to purchase securities of the first type, then those of the second type, and so on until the residual financial resources

become insufficient to purchase a lot of securities of type l in the amount V_l. In this case, the integer constraints on the purchase of stocks of type l are not considered: the maximum possible quantity of such securities is purchased. This quantity V_l' is given by

$$V_l' = \frac{F_{l-1}}{\alpha_l}, \tag{1.7}$$

where F_{l-1} is the balance of the money after purchasing the first $(l-1)$ lots of securities ($l = \overline{1,n}$).

As a result, the upper estimate is calculated as

$$Z^{\text{upp}} = \sum_{i=1}^{l-1} V_i \bar{\gamma}_i - \sum_{i=1}^{l-1} V_i \alpha_i + V_l' (\bar{\gamma}_l - \alpha_l). \tag{1.8}$$

2. Calculation of the lower estimate. The lower estimate is calculated as

$$Z^{\text{low}} = \sum_{i=1}^{l-1} V_i \bar{\gamma}_i - \sum_{i=1}^{l-1} V_i \alpha_i + F_{l-1}. \tag{1.9}$$

Upon calculating the upper and lower estimates of the optimal portfolio return, all possible portfolios are examined, and the current upper estimates for the next admissible portfolio are determined. These estimates are used to discard obviously nonoptimal solutions.

3. Calculation of the current upper estimates. When examining the next possible portfolio, the current upper estimate is calculated each time after allocating the financial resources to purchase the next lot. This estimate consists of the profit from purchasing the securities on the allocated money and the profit from the remaining securities obtained by the rule Z^{upp}. If $Z_{\text{cur}}^{\text{upp}} \le Z^{\text{low}}$, this portfolio is not considered.

Otherwise, the next lot of stocks is included in the portfolio, and $Z_{\text{cur}}^{\text{upp}}$ is calculated again. Here $Z_{\text{cur}}^{\text{upp}}$ denotes the current upper estimate. As a result, either the examined portfolio will be rejected or the portfolio with a return higher than Z^{low} will be formed. In this case, we take the objective function value on the last portfolio as the lower estimate and proceed to examining the next possible portfolio. The algorithm terminates either after enumerating all possible portfolios (in this case, the optimal solution will be the one with the last value Z^{low}) or after obtaining the portfolio with the objective function value Z^{upp}.

The problem of reliably forecasting prices γ_i arises during the practical use of the proposed solution. If we know the distribution function of the random variables describing the possible profit for each type of security, we choose the portfolio maximizing the expected gain or minimizing the risk of financial loss (the standard deviation).

Another approach to this problem with the inaccurate forecasting of future asset prices is to analyze the stability of the solution to variations of γ_i. The stability of problem (1.1)–(1.5) will be understood as the estimated effect of variations in the future prices of securities on its solution and the objective function. Three approaches are possible as follows.

In the first case, the minimum values of the prices γ_i are assumed known. Then it is required to calculate the maximum increase in these values without violating the optimal solution of the problem. In other words, it is necessary to determine ε^m such that the solution will remain the same when all γ_i increase by any $\varepsilon \in (0, \varepsilon^m)$. Here ε^m is the right limit of the range of ε.

Let the set X^j, $j = \overline{1,N}$, where N denotes the number of admissible portfolios, be the set of all possible solutions of the problem. Assume that they are arranged in ascending order of value

$$W_q = \sum_{i=1}^{n} x_i^q V_i, \quad q = \overline{1,n}. \tag{1.10}$$

Suppose that the vector x_i is optimal without perturbations. The optimal solution may change with increasing γ_i by perturbation ε. In this case, new solutions can be only the ones with numbers exceeding l. For the optimal solution x^l, the right limit of the range of ε is calculated from the relation

$$\varepsilon^l = \min_{k=l+1,n}\left\{\sum_{i=1}^{n} x_i^l V_i(\gamma_i+\varepsilon) = \sum_{i=1}^{n} x_i^k V_i(\gamma_i+\varepsilon)\right\}. \tag{1.11}$$

We solve Eq. (1.9) by removing the parentheses and expressing ε through the parameters V_i, γ_i, x_i^l, and x_i^k:

$$E^l = \max_{k=l+1,n}\frac{\sum_{i=1}^{n} x_i^k V_i\gamma_i - \sum_{i=1}^{n} x_i^l V_i\gamma_i}{\sum_{i=1}^{n} x_i^k V_i - \sum_{i=1}^{n} x_i^l V_i}. \tag{1.12}$$

Let this minimum be reached at some $l_1 > l$. Then the procedure of increasing ε^{l_1} for the solution x^{l_1} is repeated. In a finite number of steps, the solution from the set X with the maximum upper index will become optimal. And the further increase of all values γ_i will not yield a new solution.

In the second case, assume that under perturbations, the prices γ_i change according to the rule $\gamma_i + m_i\varepsilon$. In this case, the general considerations remain the same, but the solutions are arranged in ascending order of value

$$W_q = \sum_{i=1}^{n} x_i V_i m_i.$$

The increase ε^l under which the optimal solution x^l will remain the same is calculated as

$$\varepsilon^l = \min_{k=l+1,n}\frac{\sum_{i=1}^{n} x_i^k V_i\gamma_i - \sum_{i=1}^{n} x_i^l V_i\gamma_i}{\sum_{i=1}^{n} x_i^k V_i m_i - \sum_{i=1}^{n} x_i^l V_i m_i}. \tag{1.13}$$

In the third case, assume that the price γ_i takes arbitrary values from the range $[\gamma_i^1, \gamma_i^2]$. In this case, the set where the values $\gamma = (\gamma_1, .., \gamma_n)$ change can be divided into subsets $S_1, \dots, S_k$. If γ changes on any of the subsets S_j, $j = \overline{1,k}$, the optimal solution on this subset will be $x^j \in X$.

Consider the problem with $\gamma_i \in [\gamma_i^1, \gamma_i^2]$; i.e., the future expected price of asset *i* takes an arbitrary value from the range $[\gamma_i^1, \gamma_i^2]$. In this case, generally speaking, it is impossible to uniquely arrange all assets in descending order of the return. Therefore, we can form all admissible portfolios and then calculate F_j^1 and F_j^2, $j = \overline{1,N}$, for each portfolio. Here *N* denotes the number of possible portfolios, whereas F_j^1 and F_j^2 are the values of the objective function (1.6) under the minimum and maximum future prices of asset *i*, respectively. Next, we arrange the corresponding values of the objective function on the return axis for different investment portfolios.

Let us choose portfolios that can be optimal for certain future prices of their assets. For this purpose, from the set of all possible portfolios *N*, we select those satisfying the following conditions:

(1) $\max F_j^2 = F_l^2$, $j \in N$;

(2) $\max F_j^1 = F_k^1$, $j \in N$;

(3) $F_l^2 \le F_k^1$.

We denote by N_1 the remaining set of portfolios. Obviously, only portfolios from the set N_1 can be optimal under the future asset prices $\gamma_i \in [\gamma_i^1, \gamma_i^2]$, $i = \overline{1,n}$. For each admissible portfolio *j*, the value of the objective function can be written as

$$\sum_{i=1}^{n}\gamma_i x_i^j V_i - \sum_{i=1}^{n}\alpha_i x_i^j V_i + F, \tag{1.14}$$

where the Boolean vector $x^j = (x_1^i,\ldots,x_2^j)$ specifies the lots included in portfolio j.

Obviously, the set of future asset prices under which portfolio j will be optimal is given by the system of linear inequalities

$$\gamma_i^1 \le \gamma_i \le \gamma_i^2, \quad i = \overline{1,n},$$

$$\sum_{i=1}^{n}(\gamma_i - \alpha_i)x_i^j \ge \sum_{i=1}^{n}(\gamma_i - \alpha_i)x_i^l, \quad l \in N_1, \quad l \ne j. \tag{1.15}$$

Below, we consider integer modifications of portfolio investment models with risk.

Note that the proposed approaches for assessing the stability of optimization models to form investment portfolios are original. They can serve both for further theoretical research and applications. In the latter case, due to the stability domain assessment, repeated calculations can be avoided for the optimization model under local variations of the model parameters. Since the optimization model is NP-hard, this will reduce the response time of the decision maker to changes in the environment.

2. AN INTEGER CHOICE OPTIMIZATION MODEL WITH RISK

Consider an integer CAPM model. Its continuous modification was presented [1]. To determine the optimal integer portfolio of this model, we propose a branch-and-bound scheme. Such a method is needed: the transition from a continuous solution to an integer solution by rounding may cause a significant loss of accuracy.

Assume that the list of lots containing securities of the same type is known. The volume of securities (the quantity of stocks of each type) is given by values $V_1, V_2, \ldots, V_n$. Also, we know the initial prices α_i of all stocks at the time instant $t = 0$ and the probability distribution of the future prices of all stocks at the time instant $t = T$.

Let β_i, $i = \overline{1,n}$, be the coefficients for each type of financial asset. These coefficients are the quantitative assessment of risk for each type of securities. Under these conditions, the analyst with a limited budget F seeks to purchase lots that will maximize the expected increase ΔF in his financial resources by selling at the time instant $t = T$ under the risk constraints on the portfolio.

We formulate an optimization problem for determining the investment portfolio under the assumptions given above. Let the future price of asset i be described by the distribution $\gamma_i^l,\ldots,\gamma_j^m$ with probabilities $p_i^l,\ldots,p_j^m$. Then the expected future price of asset i is

$$\overline{\gamma}_i = \sum_{j=1}^{m}\gamma_i^j p_i^j.$$

In these notations, the corresponding optimization problem of the choice of the investment portfolio can be written as follows:

$$\sum_{i=1}^{n}V_i x_i(\overline{\gamma}_i - \alpha_i) \to \max, \tag{2.1}$$

$$\sum_{i=1}^{n}V_i x_i \alpha_i \le F, \tag{2.2}$$

$$\sum_{i=1}^{n}V_i x_i \alpha_i \frac{\beta_i}{F} \le \beta_{\lim}, \tag{2.3}$$

$$x_i \in \{0,1\}, \quad i = \overline{1,n}. \tag{2.4}$$

Here $\beta_{\lim}$ is the limit value determining the maximum acceptable level of risk for the portfolio.

In problem (2.1)–(2.4), $x_i = 0$ if the lot V_i is not included in the investment portfolio, and $x_i = l$ otherwise. To obtain the optimal solution of problem (2.1)–(2.4), we have to choose lots from the set $V_i,\ldots,V_n$ maximizing the objective function (2.1) subject to constraints (2.2)–(2.4).

To solve this problem, we develop the following branch-and-bound scheme.

Step 1. Calculating the upper estimate F_{upp} of problem (2.1)–(2.4). For this, we replace constraint (2.4) in (2.1)–(2.4) with

$$0 \le x_i \le 1, \quad i = \overline{1,n}. \tag{2.5}$$

Then problem (2.1)–(2.3) and (2.5) becomes a continuous linear programming problem, and its optimal solution can be obtained using, e.g., the simplex method.

We denote by x^{opt} the solution of problem (2.1)–(2.5). The objective function (2.1) on this solution is taken as F_{upp}. Here, x^{opt} is the optimal solution to the problem. Generally speaking, x^{opt} is not an admissible solution of the original problem (2.1)–(2.4): it may be not an integer. Clearly, the value of the objective function (2.1) of problem (2.1)–(2.4) on the optimal solution will not exceed F_{upp}.

Step 2. Calculating the lower estimate F_{low} of problem (2.1)–(2.4). For this, we choose some admissible solution of problem (2.1)–(2.4). The objective function (2.1) on this solution is taken as F_{low}. Note that the closer F_{low} is to F_{upp}, the more effectively the scheme will work in the future; if $F_{\text{low}} = F_{\text{upp}}$, the solution chosen above will be optimal. If $F_{\text{low}} < F_{\text{upp}}$, we proceed to the next step.

Step 3. Analyzing the current upper estimates of the portfolio.

If $F_{\text{low}} < F_{\text{upp}}$, we form the next portfolio. During this process, the current upper estimates are calculated:

$$F_{\text{upp}}^{\text{cur}}(K) = \sum_{i \in K} V_i \overline{\gamma}_i + F_{\text{upp}}. \tag{2.6}$$

Here K denotes the set of lots included in the portfolio; N and $N \backslash K$ are the sets of all lots and unpurchased lots, respectively; F_{upp} is the upper estimate of problem (2.1)–(2.4) on the set $N \backslash K$ and the volume of financial resources

$$F_K = F - \sum_{i \in K} \alpha_i V_i.$$

Further formation of the next portfolio occurs only under the conditions

$$F_{\text{upp}}^{\text{cur}}(K) > F_{\text{low}}, \tag{2.7}$$

$$\sum_{i \in K} (V_i \alpha_i \beta_i)/F \le \beta_{\text{lim}}. \tag{2.8}$$

If at least one of constraints (2.7) and (2.8) fails, we form another portfolio. If (2.7) and (2.8) hold, we select the next lot to include in the portfolio and obtain the set of purchased lots K_l. Obviously, $K \in K_l$.

On the set K_l, we calculate $F_{\text{upp}}^{\text{cur}}(K_l)$ (2.6) and check conditions (2.6) and (2.7). Continuing this procedure, we arrive at the following result: either the formed portfolio is rejected or the balance of financial resources becomes insufficient to purchase additional lots. In this case, we calculate the objective function (2.1) on the resulting admissible solution, denoting this value by F^*. If $F^* > F_{\text{low}}$, let $F_{\text{low}} = F^*$, and we move to the next investment portfolio. The calculations terminate if the next correction of F_{low} yields $F_{\text{low}} = F_{\text{upp}}$ or all possible portfolios have been considered. Then the optimal portfolio corresponds to the last (largest) value F_{low}.

3. THE MARKOWITZ INTEGER OPTIMIZATION MODEL WITH THE MINIMUM PORTFOLIO RISK CRITERION

In contrast to the classical Markowitz model [1], let the assets be purchased in lots only. We introduce the value

$$d_i = \frac{V_i \alpha_i}{F},$$

which specifies the share of investment in the given type of assets.

In the adopted notations, the Markowitz problem on the minimum risk is written as follows:

$$\sum_{i=1}^{n}\sigma_i^2 d_i^2 y_i^2 + 2\sum_{i=1}^{n}\sum_{j>i} y_i y_j d_i d_j R_{ij} \to \min, \tag{3.1}$$

$$\sum_{i=1}^{n} y_i V_i \alpha_i \le F, \tag{3.2}$$

$$\sum_{i=1}^{n} y_i V_i \overline{\gamma}_i + \left(F - \sum_{i=1}^{n} y_i V_i \alpha_i\right) \ge F + \Delta F, \tag{3.3}$$

$$y_i \in \{0,1\}, \quad i = \overline{1,n}, \tag{3.4}$$

where σ_i is the standard deviation, σ_i^2 is the variance, d_i is the share of investment in a given type of assets, and $R_{ij} = \operatorname{cov}(i,j)$ is the cross covariance of the returns of assets i and j.

In problem (3.1)–(3.4), $y_i = 1$ if lot i is included in the portfolio, and $y_i = 0$ otherwise. The value ΔF specifies the minimum possible increment of investment resources when selling the portfolio assets at the time instant $t = T$.

We describe directed enumerative search implementing the branch-and-bound scheme with our algorithms for calculating the upper, lower, and current lower estimates for this problem.

Step 1. Calculate the upper estimate for the optimal value of the objective function (3.1). To do it, we solve the auxiliary problem

$$\sum_{i=1}^{n} y_i V_i \overline{\gamma}_i + \left(F - \sum_{i=1}^{n} y_i V_i \alpha_i\right) \to \max, \tag{3.5}$$

$$\sum_{i=1}^{n} y_i V_i \alpha_i \le F, \tag{3.6}$$

$$y_i \in \{0,1\}, \quad i = \overline{1,n}. \tag{3.7}$$

Problem (3.5)–(3.7) is a linear programming problem with binary variables, and its objective function value on the optimal solution will be the upper estimate F_{upp} of problem (3.1)–(3.7).

We denote by y^{opt} the objective function value of problem (3.5)–(3.7) on the optimal solution. If this value is smaller than $F + \Delta F$, the right-hand side of (3.3), then problem (3.1)–(3.4) has no solution. Otherwise, we calculate the value of the objective function (3.1) on this solution, taking it as the upper estimate R_{upp} for problem (3.1)–(3.4).

Step 2. Choosing zero as the lower estimate: $R_{\text{low}} = 0$. In the case $R_{\text{low}} < R_{\text{upp}}$, we proceed to Step 3. If $R_{\text{low}} = R_{\text{upp}}$, the optimal solution of the original problem is found.

Step 3. Calculating the current lower estimates for possible portfolios. Assume that the portfolio includes lots of the set K, and

$$\sum_{i\in K} y_i V_i \alpha_i \le F.$$

The current lower estimates for possible portfolios are calculated as follows.

First, we arrange all lots of the set $N\backslash K$ by the return on assets:

$$\overline{\gamma}_1/a_1 \ge \overline{\gamma}_2/a_2 \ge \ldots \ge \overline{\gamma}_n/a_n. \tag{3.8}$$

Then we check the inequality

$$\sum_{i\in K} y_i V_i \gamma_i + F_{\text{upp}}(N\backslash K) \ge F + \Delta F. \tag{3.9}$$

If (3.9) holds, we choose the next lot from the set $N\backslash K$. Including it in the portfolio and forming the set of lots $K_l\,(K \in K_l)$, we calculate the current lower estimate for the lots from the set K_l.

The portfolio formation process terminates in one of the following cases: condition (3.9) is violated by including a new lot in the portfolio, or the residual financial resources are insufficient to purchase at least one of the remaining lots. In the latter case, we check the value of the objective function (3.1) on the portfolio. If it is smaller than R_{upp}, we take R_{upp} as the value of the objective function (3.1).

The branch-and-bound scheme stops if the next correction of R_{upp} gives $R_{\text{upp}} = R_{\text{low}}$ or the entire tree of possible portfolios is exhausted. Then the optimal portfolio corresponds to the last (smallest) value R_{upp}.

4. THE INTEGER MODEL WITH THE MAXIMUM RETURN CRITERION AND A RISK CONSTRAINT

Consider the integer Markowitz model with the maximum expected return criterion and a risk constraint on the securities portfolio. In the adopted notations, it can be written as

$$\sum_{i=1}^{n} y_i V_i (\overline{\gamma}_i - \alpha_i) + F \to \max, \tag{4.1}$$

$$\sum_{i=1}^{n} y_i \sigma_i^2 d_i^2 + 2\sum_{i=1}^{n}\sum_{j>i} y_i y_j d_i d_j R_{ij} \le R, \tag{4.2}$$

$$\sum_{i=1}^{n} y_i V_i \alpha_i \le F, \tag{4.3}$$

$$y_i \in \{0,1\}, \quad i = \overline{1,n}. \tag{4.4}$$

We solve the integer problem (4.1)–(4.4) using the branch-and-bound scheme.

Step 1. Calculating the upper estimate for the optimal value of the objective function in problem (4.1)–(4.4). To do it, we eliminate (4.2) and replace (4.4) by

$$0 \le y_i \le 1, \quad i = \overline{1,n}. \tag{4.5}$$

In this case, the maximum return on the portfolio in problem (4.1)–(4.3) and (4.5) can be determined through ordering lots by the ratio $\overline{\gamma}_1/a_1$, $i = \overline{1,n}$; see above.

We renumber the lots in descending order of $\overline{\gamma}_1/a_1$ and find $\overline{\gamma}_1/a_1 \ge \overline{\gamma}_2/a_2 \ge \ldots \ge \overline{\gamma}_n/a_n$. Then we purchase lots in descending order of $\overline{\gamma}_1/a_1$ until all financial resources F are exhausted. Clearly, this portfolio will be the optimal solution of problem (4.1)–(4.3), (4.5).

If the resulting portfolio also satisfies constraints (4.2) and (4.4), it will be a solution of the original problem (4.1)–(4.4). If this condition fails, we proceed to Step 2.

Step 2. Calculating the lower estimate for the optimal value of the objective function. As the lower estimate F_{low} of problem (4.1)–(4.4), we can take the value of its objective function on some admissible solution.

Step 3. Calculate the current upper estimates of the optimal value of the objective function to form the investment portfolio. Assume that the portfolio includes lots of the set K. The current upper estimates for possible portfolios are given by

$$F_{\text{upp}}^{\text{cur}}(K) = \sum_{i\in K} \overline{y}_i V_i + F_{\text{upp}}. \tag{4.6}$$

Here F_{upp} denotes the upper estimate of problem (4.1)–(4.4) on the set of lots $N \backslash K$.

Upon calculating the values $F_{\text{upp}}^{\text{cur}}(K)$, we check the inequality

$$F_{\text{upp}}^{\text{cur}}(K) > F_{\text{low}}. \tag{4.7}$$

If (4.7) holds, the next lot purchase is selected, and an investment portfolio is formed with the set of lots $K(K \in K_l)$. If inequality (4.7) holds on the entire set K_l, the portfolio formation process continues. Otherwise, this portfolio is rejected, and a new investment portfolio is formed.

If this procedure yields a portfolio satisfying all the constraints (4.2)–(4.4), and the value of the target function F^* on it is greater than F_{low}, we take $F_{\text{low}} = F^*$ and form a new investment portfolio.

The final step is when, after the next correction of F_{low}, we obtain $F_{\text{low}} = F_{\text{upp}}$ or all possible portfolios are exhausted. In this case, the optimal portfolio corresponds to the last (largest) value F_{low}.

The two-criteria integer optimal choice models considered in Sections 2–4 and our methods for solving them are original. They allow optimizing securities portfolios with indivisible assets.

5. INTEGER OPTIMAL CHOICE WITH RETURN AND RISK CRITERIA: AN ALTERNATIVE PROBLEM STATEMENT AND THE SOLUTION METHOD

Consider an integer modification of the Markowitz model with unknown numbers of lots of the issuer's securities included in portfolios that differ by return, risk, and investment budget. (The problem statement with unknown lots of homogeneous risky financial assets included in the investment portfolio was presented, e.g., in [17].)

Such modification of the optimal portfolio investment problem requires an appropriate correction of the variables and parameters. In the discrete Markowitz model, the desired variables are x_i (the number of lots of stock i in the portfolio). The investment S_i in stock i is given by

$$S_i = x_i c_i. \tag{5.1}$$

Here c_i denotes the purchase (bid) price of stock i.

Recall that assets can be purchased in lots only. Therefore, the shares of investment in the portfolio are calculated as

$$W_i = \frac{x_i c_i}{S_o}, \tag{5.2}$$

where S_o denotes the investor's budget.

We introduce the change $x_i c_i = \gamma_i$ (the investor's budget to purchase the lots of stocks i):

$$\sum_{i=1}^{n} \gamma_i d_i \to \max. \tag{5.3}$$

$$\sqrt{\frac{\sum_{i=1}^{n} \gamma_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{i+1}^{n} \gamma_i \gamma_j r_{ij} \sigma_i \sigma_j}{\sum_{i=1}^{n} \gamma_i^2}} < \sigma, \tag{5.4}$$

$$\sum_{i=1}^{n} \gamma_i \le S_0, \tag{5.5}$$

$$\gamma_i \ge 0, \tag{5.6}$$

where d_i is the average return on asset i; σ_i is the standard deviation of the return on asset i; r_{ij} is the correlation coefficient of the returns on assets i and j; and σ is the portfolio risk threshold.

The formal problem statement (5.3)–(5.6) allows reducing the volume of calculations: we find the amounts γ_i allocated to the purchase of securities instead of the share W_i of the issuers' stocks. However, it neglects the discrete nature of purchased lots. We therefore restrict the variables x_i to integers. (Recall that x_i is the quantity of purchased lots of stocks i). As a result, the model with discrete purchased lots takes the form

$$\sum_{i=1}^{n} x_i c_i d_i \to \max, \tag{5.7}$$

$$\sqrt{\frac{\sum_{i=1}^{n} x_i^2 c_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{i+1}^{n} x_i c_i x_j c_j r_{ij} \sigma_i \sigma_j}{\sum_{i=1}^{n} x_i^2 c_i^2}} < \sigma, \tag{5.8}$$

$$\sum_{i=1}^{n} x_i c_i \leq S_0, \tag{5.9}$$

$$x_i c_i \geq 0, \tag{5.10}$$

$$x_i \in Z_+. \tag{5.11}$$

Problem (5.7)–(5.11) is an integer nonlinear programming problem belonging to the class of *NP*-complete Turing problems. Such problems, see the discussion above, cannot be solved numerically by known nonlinear continuous optimization methods. This aspect was described in detail, e.g., in [18].

However, in our case, the risk constraint (5.8) is convex (a quadratic form), and criterion (5.7) is linear. Therefore, we involve the main idea of the branch-and-bound scheme to find the optimal solution of the problem. This idea consists in representing the domain of admissible values as a direct sum of appropriate nonintersecting convex domains: for each of them, the optimal solution of the integer problem is obtained by the well-known optimization methods (e.g., reduction to a linear problem).

We propose a method to find not the optimal but quasi-optimal solution of the discrete problem with the maximum portfolio return criterion. This method is called the local optimization method for the solution of the corresponding continuous problem.

The numerical algorithm includes the following steps [19].

Step 1. The problem is solved in the continuous statement (with infinitely divisible assets). The assets in the resulting solution are arranged in descending order of return.

Step 2. The values x_i are randomly rounded up or down, and the admissibility of the solution (the constraints on the investment budget and risk threshold) is checked. If the solution is inadmissible, the random experiment is repeated. Finally, an admissible integer portfolio is obtained, and the algorithm proceeds to Step 3.

Step 3. A sphere of unit radius is constructed around the admissible plan. Possible portfolios differing from the basic portfolio (Step 1) by one for each asset in it are considered. They are arranged in descending order of return until the resulting set of assets becomes admissible. The best admissible (quasi-optimal) portfolio is subjected to the same procedure again.

The iterative algorithm terminates if the newly obtained portfolio differs from the previous one at most by τ percent, where τ is the given accuracy threshold.

According to [19], this threshold grows with the number of financial assets in the portfolio. The greater the number of securities the higher the accuracy will be. (The cited work considered a portfolio of possible production programs of an enterprise, including products of assortment range, with interval-type prices and cost determined by the market. As shown there, in the case of a production program with more than 100 products (assets), the error of the quasi-optimal solution is 5–7% compared with the optimal solution of the integer problem obtained by a simple enumerative algorithm; moreover, the error decreases with increasing the portfolio size).

Integer modifications of the Markowitz model retain the main features of the classical formulation, except, of course, the infinite divisibility of assets. The integer models allow considering the effect of discreteness, initial budget, and liquidity on the optimal structure of the investor's portfolio obtained by the classical model. In practical calculations, we estimated the effect of these factors on the portfolio structure based on the Moscow Exchange data for the period from March 1, 2021, to November 20, 2021; see the paper [20].

CONCLUSIONS

This paper has proposed methods to assess integer models of limited resource management. They are used in the case of indivisible assets in the portfolio. Therefore, for such models, the classical approaches based on the divisibility of assets are unacceptable. The proposed methods are relevant, in particular, when selecting a set of indivisible projects, making wholesale purchases of material production resources, or forming portfolios of indivisible financial and tangible assets. As demonstrated, the optimal portfolio by the risk-return criteria can be formed using integer optimization methods, particularly the branch-and-bound scheme developed above. In the case of interval-type future prices of assets, the stability analysis method has been proposed for the optimal portfolio.

CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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