Dynamic models of production-finance activity of enterprises in conditions of uncertainties and risk

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Abstract: The problem of optimal management of various types of production resources in conditions of uncertainty and risk taking into account recoverable and unrecoverable defects is relevant and necessary to ensure the competitiveness of modern industrial enterprises. The aim of the research is to develop mathematical tools for evaluating the effectiveness of industrial logistics systems, taking into account uncertainty and risk. The paper presents a set of dynamic optimisation models for evaluating the effectiveness of industrial logistics systems, the solution of which is a vector function of time, which sets the productivity of production operations according to the criterion of profit maximisation, taking into account a number of constraints. The results of this work are the development of: the complex of optimal control models taking into account uncertainty and risk; the tool to convert an optimal control problem to linear programming model; the evaluation mechanism of the stability of solutions under changes in the prices of end products; risk assessment methods of the obtained solution. The practical use of the developed tools was shown in evaluating the effectiveness of the investment program of the manufacturer of computers.

Keywords: optimisation models; production activity; uncertainty; recoverable and unrecoverable defects.

Reference to this paper should be made as follows: Mishchenko, A.V., Solodovnikov, V.V. and Apal’kova, T.G. (2022) ‘Dynamic models of production-finance activity of enterprises in conditions of uncertainties and risk’, *Int. J. Logistics Systems and Management*, Vol. 43, No. 1, pp.86–111.

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1 Introduction

The problem of optimal management of various types of production resources in conditions of uncertainty and risk taking into account recoverable and unrecoverable defects is relevant and necessary to ensure the competitiveness of modern industrial enterprises.

Mathematical models are widely used in analysis of logistics systems (Chanchaichujit et al., 2019). For example, such models are used in the works of Mishchenko (2017), Mittal et al. (2017), Kocaoglu and Acar (2016), Goyal et al. (2013), etc. In contrast to these works, the article deals with dynamic models of material and production resources management in industrial logistics which take into account uncertainty (recoverable and unrecoverable defects) and risk.

The aim of the research is to develop mathematical tools for evaluating the effectiveness of industrial logistics systems, taking into account uncertainty (recoverable and unrecoverable defects) and risk. The paper presents a set of dynamic optimisation models for evaluating the effectiveness of industrial logistics systems, the solution of which is a vector function of time, which sets the productivity of production operations according to the criterion of profit maximisation, taking into account a number of constraints.

Scientific survey is based on fundamental and applied developments of domestic and foreign scientists in the field of economic theory, management theory, logistics theory and supply chain management, methods of mathematical modelling, system analysis, operations research and expert evaluation methods.

The dynamic optimisation models are developed in the paper according to scientific approach: moving from simple to complex (logical method). The first part of the study presents single-period formulation of problem. Next, multi-period models of production and financial activity of the enterprise are considered as the development of single-period models. Finally, models of management of production and financial activities of the enterprise, taking into account the risk are considered as the most general and complex form.

2 Literature review

The theory and practice of dynamic optimisation models are represented with different level of details in contemporary literature.

Sergeev and Solodovnikov (2020) proposed a general approach to the development of industry methodology of integrated supply chain planning. Elleuch and Frikha (2020) discussed the issue of selecting a set of potential facility sites. Mashud (2020) presents a deteriorating EOQ inventory model according to consideration of the price, stock dependent demand and fully backlogged shortages. Mageto et al. (2020) provide overview of logistics outsourcing performance and their relationship with logistics performance among manufacturing small and medium enterprises. Ishtiaque et al. (2020) presents a model that facilitates an understanding of relationships among information and communication technology, integrative capabilities, operational responsiveness and dimensions of performance in a developing economy. Gandhi et al. (2020) investigates the role played by service quality in the supply chains of small-medium manufacturing units and presents models at different junctions, which propose and validate that the contributions made by supply chain partners towards service quality lead to satisfaction and loyalty. Le et al. (2019) address the distribution network design problem in a complex four-echelon supply chain system that includes factories, internal warehouses, external warehouses and customers.

As it can be seen from works of Berezhnaya (2006), Burkov (2001), Plotnikov (2006), Taha (2005) and Winston (1991) various methods and models of quantitative analysis can be used for assessing the effectiveness of management of production enterprise resources. Currently, various tools has been developed for such analysis according to Voronin (2008), Shapiro (2006), Shimko (2004), Ivert (2012), Kambo (1984), Sergeev (2015) and Stadtler (2004).

Chanchaichujit et al. (2019) map and define the modelling approaches and mathematical techniques that have been used in solving supply chain problems in various industries.

Buyurgan et al. (2019) present two compensations methods for inventory discrepancy caused by demand, supply and lead time uncertainty as well as inventory related error.

Quang and Hara (2019) propose and validate a conceptual framework for linking various dimensions of risk to system performance in the supply chain by applying supply chain mapping.

Mishchenko (2017) developed dynamic management model of current assets for a trading company.

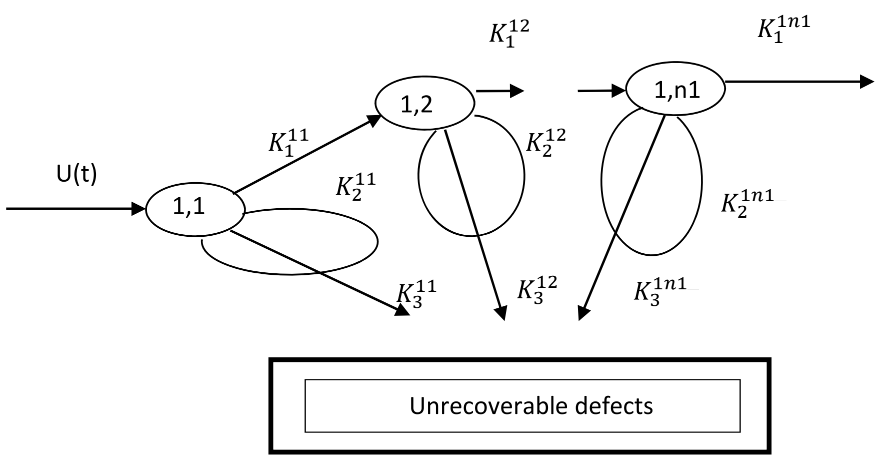
In contrast to these works, the article deals with dynamic models of material and production resources management in industrial logistics.

3 Main part

3.1 Single-period models of optimisation of production and financial activity of an enterprise

It is assumed that the enterprise produces N types of products and the production technology of each type of products consists of the sequential processing of materials and raw materials. This sequence is given by the approximate graph $G(m, n)$ (Figure 1).

Figure 1 Structural diagram of technological operations of the product, taking into account the recoverable and unrecoverable defects



$U_1(t) = (U_{11}(t), \ldots, U_{1M}(t))$ (Figure 1) sets the flow of material and raw resources sent to the input of the technological production chain.

The coefficient K_1^{1j} $(j = 1, 2, \ldots, n_1)$, sets the share of material resources from the total amount of material resources processed at operation j, which is passed to the operation $j + 1$.

The coefficient K_2^{1j} determines the share of material resources from the total amount of material resources processed at operation j, which is a recoverable defect. This part of material resources after additional processing at one of the previous operations returns to the production cycle. This is a recoverable defect.

The coefficient K_3^{1j} $(j = 1, 2, \ldots, n_1)$, sets the share of material resources, which, after processing at operation *j*, is derived from the production cycle and is not returned to production later. This is an unrecoverable defect.

If we consider a production system that produces *N* types of products, then we will assume:

$$K_1^{1j} \geq 0, K_2^{1j} \geq 0, \sum_{l=1}^{3} K_l^{1j} = 1; \forall i = 1, 2, \ldots, N, \ j = 1, 2, \ldots, n_i.$$

The intensity of the output of finished products is denoted by $q_{1n_1}(t)K_1^{n_1}$, and accordingly, the volume of production of the first type product in the time interval (0, *T*) is equal to:

$$W_1(t) = \int_O^T K_1^{1n_1} q_{1n_1}(t)\mathrm{d}t \tag{1}$$

In equation (1), it is assumed that if $K_1^{1n_1}$ is a random value, then its mathematical expectation is used, i.e.,

$$K_1^{1n_1} = \sum_{q=1}^{m} K_{1q}^{1n_1} P_{1q}^{1n_1}$$

Here, $P_{iq}^{1n_1} \geq 0$ is a probability that $K_1^{1n_1} = K_{1q}^{1n_1}$ $\forall q = 1, 2, \ldots, m; \sum_{q=1}^{m} P_{1q}^{1n_1} = 1.$

One of criteria for optimising the production system is the amount of profit received from the sale of the final output produced in the interval (0, *T*). Let *N* types of products be produced and the marginal income from the sale of one unit of production is defined as:

$$\beta_i = a_i - b_i, i = 1, 2, \ldots, N.$$

Here, a_i is the sales price for products of the type *i*, b_i – variable costs associated with the production of products of the type *i* (*i* = 1, 2, …, *N*).

Then, the marginal income from the release of *N* types of products in the interval (0, *T*), which must be maximised, is given as follows:

$$\sum_{i=1}^{N} \beta_i \int_O^T K_1^{in_i} q_{in_i}(t)\mathrm{d}t \to \max \tag{2}$$

At the same time, of course, it is necessary to take into account the demand constraints for each type of products: constraints on the production capacity of an enterprise, on which the productivity of the work in progress at the operations O_{ij} (*i* = 1, 2, …, *N*, *j* = 1, 2, …, n_i) is dependent; constraints related to the fact that the volume of output must ensure the fulfilment of the order, as well as constraints on the volume of purchases of material resources of production.

These constraints will be formulated below:

$$\begin{gathered} V_{ijk}(0) + \int_o^t K_1^{ij-1} q_{ij-1k}(t')dt' + \sum_{Opg \in Rij} \int_0^t K_2^{pg} q_{pg}(t')dt' \geq \int_o^t q_{ijk}(t')dt' \\ \forall t \in (0, T) \ i = 1, 2, \ldots, N; \ j = 1, 2, \ldots, n_i; \ k = 1, 2, \ldots, M. \end{gathered} \tag{3}$$

Here, R_{ij} is a set of operations, from which unrecoverable defects enters the operation O_{ij}. The constraint (3) indicates that the volume of processing of work in progress at the operation O_{ij} ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, n_i$) for each time interval $(0, t) \le (0, T)$ cannot be more than the volume of work in progress at time $t = 0$ [value $V_{ijk}(0)$] plus the volume of work in progress, which came from the previous operation O_{ij-1} taking into account the coefficient K_1^{ij-1}, plus the amount of unrecoverable defects received for the operation O_{ij}.

$$\sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{q_{ijk}(t)}{q_{ijk}^{o}(t)} \alpha_{ijl} \le C_l \;\; l = 1, 2, \ldots, n$$
$$\forall t \in (0, T) \quad i = 1, 2, \ldots, N;\; j = 1, 2, \ldots, n_i;\; k = 1, 2, \ldots, M \tag{4}$$

$$\int_O^T K_1^{ini} q_{ini}(t)dt \ge Zak_i \quad i = 1, 2, \ldots, N \tag{5}$$

$$\int_O^T K_1^{ini} q_{ini}(t)dt \le Pt_i \quad i = 1, 2, \ldots, N \tag{6}$$

$$q_{ijk}(t) \ge 0;\; \forall t \in (0, T);\; i = 1, 2, \ldots, N;\; j = 1, 2, \ldots, n_i;\; k = 1, 2, \ldots, M. \tag{7}$$

Taken into account constraint (5), tasks (1)–(7) is not always solvable due to the fact that it is impossible to ensure output of at least Zak_i or due to lack of material resources at operations O_{ij} ($i = 1, 2, \ldots, N$; $j = 1, 2, \ldots, n_i$), or due to insufficient production capacity. In the first case, it is necessary to purchase material resources of production additionally, in the second – to supply the necessary equipment additionally that will increase the productivity of the production system to the level of ratio (4) – these are constraints on the production capacity of the enterprise. Here C_l ($l = 1, 2, \ldots, m$) is the number of units of production equipment of the type 1 at the enterprise, α_{ijl} – is the number of units of equipment of the type 1 necessary to ensure the processing of material resources of all kinds in the operation O_{ij} with the lowest possible performance q_{ijk}^{o} $(k = 1, 2, \ldots, M)$. If it is necessary to ensure greater productivity at operations O_{ij}, such as for example, q_{ijk}, the volume of production resources increases by $\frac{q_{ijk}}{q_{ijk}^{o}}$ times, which allows releasing products in at least Zak_i over a period of time $(0, T)$. In this situation, it is often necessary to estimate the minimum amount of investment to ensure the planned performance of the enterprise. To solve this problem, consider the following optimisation problem:

$$\sum_{l=1}^{m} y_l \gamma_l + \sum_{i=1}^{N} \sum_{K=1}^{M} U_{i1k} W_k \to \min \tag{8}$$

Here y_l – is the number of additionally purchased units of equipment of the type l ($l = 1, 2, \ldots, m$), γ_l – is the price of a unit of equipment of the form l, U_{i1k} – is the volume of material resources of the type k supplied to the first production operation for products of the type i ($i = 1, 2, \ldots, N$), and W_k is the unit price of material resources of the form k ($k = 1, 2, \ldots, M$).

$$V_{i1k}(0) + U_{i1k} + \sum_{Opg \in R_{i1}^{o}} \int_o^t K_2^{pg} q_{pgk}(t')dt \ge \int_o^t q_{i1k}(t')dt' \tag{9}$$

Here, R_{i1} is a set of operations, from which unrecoverable defect enters the operations O_{i1}, $\forall\, t \in (O, T)$, $i_k = 1, 2, \ldots, N$;

$$V_{ijk}(0)+\int_o^t K_1^{ij-1} q_{ij-1k}(t')dt' + \sum\nolimits_{Opg\in R_{ij}^o} \int_o^t K_2^{pq} q_{pgk}(t')dt' \geq \int_o^t q_{ijk}(t')dt'; \quad (9.1)$$
$$i=1, 2, \ldots, N;\ j=1, 2, \ldots, n_i;\ k=1, 2, \ldots, M,\ \forall t \in (0, T)$$

Constraints (9) and (9.1) indicate that the volume of processing of work in progress of all types of material resources cannot be greater than the volume of received work in progress for this operation, taking into account the stock that was at operation O_{ij} at time $t = 0$ [the value $V_{ijk}(0)$]. R_{ij}^o – is the set of operations from which the unrecoverable defect is transferred to the operation O_{ij}.

$$\sum\nolimits_{i=1}^N \frac{q_{ij}(t)}{q_{ij}^o(t)} \alpha_{ijl} \leq C_l + y_l; \quad (10)$$
$$i=1, 2, \ldots, N;\ j=1, 2, \ldots, n_i;\ l=1, 2, \ldots, m;\ \forall t \in (0, T).$$

Equation (10) is a constraint on production resources, from which it follows that the volume of used production resources at each time point $t \in (0, T)$ must not exceed the value $C_l + y_l$ ($l = 1, 2, \ldots, m$).

$$\int_0^T K_1^{ini} q_{ini}(t)dt \leq Pt_i, \quad i=1, 2, \ldots, N;\ K_1^{ini} \leq 1 \quad (11)$$

Equation (11) is a constraint on the demand given by the value t_i ($i = 1, 2, \ldots, N$).

$$\int_0^T K_1^{ini} q_{ini}(t)dt \geq Zak_i, \quad i=1, 2, \ldots, N;\ K_1^{ini} \leq 1 \quad (12)$$

Equation (12) says that customer requirements should be met, i.e., products should be manufactured for the period (0, T) in the amount of not less than Zak_i ($i = 1, 2, \ldots, N$);

$$q_{ij}(t) \geq 0;\ \forall t \in (0, T);\ i=1, 2, \ldots, N;\ j=1, 2, \ldots, n_i \quad (13)$$

Tasks (8)–(13) as opposed to tasks (2)–(7) always have a solution (in the absence of constraints on the amount of investments and the possibility of purchasing). At the same time, the cost of purchasing additional equipment in the amount of y_l ($l = 1, 2, \ldots, m$) may turn out to be too large, due to the high proportion of recoverable and unrecoverable defects K_2^{ij}, K_3^{ij} $(i=1, 2, \ldots, N;\ j=1, 2, \ldots, n_i)$. In this case, along with the use of existing technology, a project to modernise the entire production can be considered, which can be implemented through the sale of old equipment and the purchase of new production facilities. This will improve the quality of products, which will lead to an increase in demand $\tilde{P}t_i$ $(\tilde{P}t_i > Pt_i;\ i=1, 2, \ldots, N)$, an increase in marginal income $\tilde{\beta}_i$ $(\tilde{\beta}_i > \beta_i;\ i=1, 2, \ldots, N)$, reducing the volume of recoverable and unrecoverable defects and, consequently, an increase in the volume of high-quality products when using the same material resources, i.e., $[\widetilde{K_1^{ij}} > K_1^{ij}, \widetilde{K_2^{ij}} < K_2^{ij}, \widetilde{K_3^{ij}} < K_3^{ij}$ $(i=1, 2, \ldots, N;.$ $j=1, 2, \ldots, n_i)]$.

In order to evaluate which of the alternatives is more efficient, it is necessary to calculate the minimum amount of investments for the implementation of the project of modernisation of the enterprise.

The following model can be used to carry out such a calculation:

$$\sum_{l=1}^{m}\tilde{y}_l\tilde{\gamma}_l+\sum_{i=1}^{N}\sum_{k=1}^{M}U_{i1k}W_k-\sum_{l=1}^{m}C_l\overline{\gamma}_l\rightarrow \min. \tag{14}$$

Here, $\tilde{y}_l$ – is respectively, the number of units of new purchased equipment, $\tilde{\gamma}_l$ – is the price of a unit of new equipment of the type l, C_l – is the number of units of equipment of the type l, previously participated in the production process, $\overline{\gamma}$ – is the residual value of the equipment of the type l ($l = 1, 2, \ldots, m$), which will be sold during the modernisation of production.

Further, similarly with equations (8)–(13), constraints could be formulated:

$$V_{i1k}(0)+U_{i1k}+\sum_{Opg\in R_{i1}^{o}}\int_{o}^{t}\widetilde{K_2^{pg}}q_{pg}(t')dt'\geq\int_{o}^{t}q_{i1k}(t')dt';\ \forall t\in(0,T) \tag{15}$$

Here, R_{i1}^{o} – is the set of operations O_{pg}, from which a recoverable defect enters operation O_{i1} ($i = 1, 2, \ldots, N; j = 1, 2, \ldots, M$)

$$V_{ijk}(0)+\int_{o}^{t}\widetilde{K_1^{ij}}q_{ij-1k}(t')dt'+\sum_{Opg\in R_{ij}^{o}}\int_{o}^{t}K_2^{pg}q_{pgk}(t')dt'\geq\int_{o}^{t}q_{ijk}(t')dt'$$
$$i=1,2,\ldots,N;\ j=1,2,\ldots,n_i;\ k=1,2,\ldots,m;\ \forall t\in(0,T) \tag{15.1}$$

Here, the set R_{ij}^{o} is the set of operations from which a recoverable defect enters operation O_{ij}.

Equations (15)–(15.1) indicate that the volume of processed material resources at operation O_{ij} cannot exceed the amount of receipt of these resources for each time interval $(o, t) \leq (0, T)$, $t \leq T$, taking into account the existing stock at the moment of time $t = 0$ ($V_{ijk}(t) \geq 0$, $i = 1, 2, \ldots, N; j = 1, 2, \ldots, n_i$).

$$\sum_{i=1}^{N}\frac{q_{ij}(t)}{q_{ij}^{o}(t)}\alpha_{ijl}\leq\tilde{y}_l;\ l=1,2,\ldots,m \tag{16}$$

Here, $\tilde{y}_l$ is the number of new units of equipment of the type l;

$$\int_{o}^{T}\tilde{K}_1^{ini}q_{ini}(t)dt\leq Pt_i;\quad i=1,2,\ldots,N;\ \tilde{K}_1^{ini}\leq 1 \tag{17}$$

It is worth mentioning that when modernising production related to upgrading existing equipment, it is assumed that $\tilde{K}_1^{ini}>K_1^{ini};\ i=1,2,\ldots N$:

$$\int_{0}^{T}\tilde{K}_1^{ini}q_{ini}(t)dt\geq Zak_i,\ i=1,2,\ldots N \tag{18}$$

$$q_{ij}(t)\geq 0;\ \forall t\in(0,T);\ i=1,2,\ldots N,\ j=1,2,\ldots n_i \tag{19}$$

Thus, solving tasks (14)–(19), we get the amount of investment in working capital and the new production system that is necessary to create a production capacity capable of carrying out output of at least Zak_i ($i = 1, 2, \ldots, N$). Further, taking into account the renewal of the production capacity of the enterprise, tasks (2)–(7) can be solved, optimising the marginal income, and based on the obtained data the net profit obtained from the sale of products released in the interval $(0, T)$ in conditions of complete replacement of production equipment can be calculated. We denote this indicator as

PR_{OPM}. Similarly, the problem of minimising investments for fulfilling an order for products (volumes not less than Zak_i) can be solved in conditions of purchasing additional equipment without replacing the already existing one [this is tasks (8)–(13)].

Further, by determining the number of equipment y_l, and the volumes of purchases of material resources U_{i1k} (l = 1, 2, …, m, i = 1, 2, …, N, k = 1, 2, …, M), the optimisation task can be solved and marginal income for this variant of the production system and the corresponding net profit can be calculated. We denote it as PR_{SPM}. Comparing PR_{OPM} and PR_{SPM}, we can draw the following conclusions:

1 If $PR_{OPM} \geq PR_{SPM}$, then it is economically reasonable to update the production system of the enterprise.

2 If when expanding the time interval (0, T) to the level (O, T'), where $T' > T$, the net profit of the updated equipment is higher than PR_{SPM}, then the choice of alternative whether to update the production unit or not depends on how much T is larger compared to T.

3.2 *Multi-period models of optimisation of production and financial activity of an enterprise*

Consider a dynamic model of management of production and financial activities of the enterprise in conditions when the period duration (0, T) is a year or more. In this situation, compared with the base models (2)–(7), it is necessary to:

1 divide the time interval (0, T) by time periods equal to one year

2 discount financial flows for each year

3 set the demand and order for each type of product for each period separately.

Taking into account the above requirements a dynamic multi-period model for optimising the production and financial activity of an enterprise can be formulated as follows:

$$\sum_{\tau=0}^{T}\left(\sum_{i=1}^{N}\beta_i^{\tau}\int_0^t K_1^{ini} q_{ini}^{\tau}(t)dt\right)\Big/(1+K)^{\tau} \rightarrow \max \tag{20}$$

Here, K is the discount rate, T is the number of time intervals, K_1^{ini} – coefficient reflecting the share of output of quality products from the O_{ini} operation, taking into account the recoverable and unrecoverable defects $(0 \leq K_1^{ini} \leq 1)$ at this operation, $q_{ini}^{\tau}(t)$ is the intensity of the processing of material resources at the operation O_{ini} at time t in the time period with the number τ. Thus, equation (20) sets the total marginal income from the sale of all types of products released for all periods of time with a number (τ = 0.1, …, T). Here, it is assumed that the value of the yield coefficient of the non-defective final production K_1^{ini} remains unchanged for all τ = 0.1, …, T.

$$\begin{gathered} V_{ijk}^{\tau}(0)+\int_o^t K_1^{ij-1} q_{ij-1k}^{\tau}(t')dt' + \sum_{Opg\in R_{ij}^o}\int_o^t K_2^{pg} q_{pgk}^{\tau}(t')dt' \geq \int_o^t q_{ijk}^{\tau}(t')dt' \\ \forall\tau = 0, 1, \ldots, T;\ \forall t \in (0, T);\ i = 1, 2, \ldots N,\ j = 1, 2, \ldots n_i \end{gathered} \tag{21}$$

Equation (21) is a balance constraint on the volume of processing of each type of material resources for each operation. In equation (21), the following notation is used: $V_{ijk}^{\tau}(0)$ is the volume of material resources of the form k at the beginning of the period with number τ for operations O_{ij}, K_1^{ij-1} is a coefficient defining the share of material resources processed without defects at operations O_{ij-1} that are passed to operation O_{ij}, $q_{ij-1k}^{\tau}(t')$ is the processing performance of a material resource of the form k at operation O_{ij-1} at time t' in the time period with number τ, R_{ij} is the set of operations from which material resources are passed to the operation O_{ij} for the elimination of an unrecoverable defects, $q_{pgk}^{\tau}(t')$ is the intensity of processing of material resources of the form k at operation O_{pg} at time t', in period number τ, K_2^{pg} is the share of a recoverable defects, which is obtained after processing material resources at the operation O_{pg}, $q_{ijk}^{\tau}(t')$ is the intensity of the processing of material resources of the type k at operation O_{ij} at the time t' of the period with the number τ.

The next restriction is the capacity constraint for each time interval with number τ.

$$\sum_{i=1}^{N}\sum_{j=1}^{m_1}\frac{q_{ijk}^{\tau}(t)}{q_{ijk}^{o}(t)}\alpha_{ijl} \le C^l$$
$$l = 1, 2, \ldots, m;\ k = 1, 2, \ldots, M;\ \tau = 1, 2, \ldots, T;\ \forall t \in (0, \tau) \tag{22}$$

The following two constraints are constraints on the order and on the volume of demand:

$$\int_{t_j}^{t_{j+1}} K_1^{ini} q_{ini}^{\tau}(t)dt \ge Zak_i^T$$
$$i = 1, 2, \ldots, N;\ j = 0, 1, 2, \ldots, T-1;\ \tau = 0, 1, 2, \ldots, T;\ \forall t \in (0, \tau) \tag{23}$$

Here, Zak_i^{τ} is the order volume for products of the type i on the time interval with the number τ. An order constraint means that there is a consumer of manufactured products who has concluded a contract with the manufacturer for the supply of finished products in the volume of the order.

$$\int_{t_j}^{t_{j+1}} K_1^{ini} q_{ini}^{\tau}(t)dt \ge Pt_i^T$$
$$i = 1, 2, \ldots, N;\ j = 0, 1, 2, \ldots, T-1;\ \tau = 0, 1, 2, \ldots, T;\ \forall t \in (O, \tau) \tag{24}$$

Here, Pt_i^{τ} is the demand for products of the type i, produced in the time period with the number τ.

$$q_{ij}^{\tau}(t) \ge 0, i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n_i;\ \tau = 0, 1, 2, \ldots, T. \tag{25}$$

3.3 Models of management of production and financial activities of the enterprise, taking into account the risk

As noted above, such parameters of the considered models as marginal revenue from the sale of a unit of production and the volume of demand can be random values, and therefore, in the calculations for the proposed models, it is necessary to take into account the risk of profitability of the chosen production program and the risks of overproduction and lost profits.

Let us return to models (2)–(7) with the assumption that it has a solution and marginal income β_i ($i = 1, 2, \ldots, N$), which is a random variable with a given discrete probability distributions, that is:

$$\beta_i \quad \begin{matrix} \nearrow \beta_i^1 - p_1 \\ * \\ \searrow \beta_i^m - p_m \end{matrix} \qquad P_j \geq 0; \sum_{j=1}^{m} P_i = 1$$

We assume that in objective function (2):

$$\beta_i = \overline{\beta}_i = \sum_{j=1}^{m} \beta_i^j P_j; \, i = 1, 2, \ldots, N$$

Let l_{ij} ($i = 1, 2, \ldots, N; j = 1, 2, \ldots, M$) denote the amount of material resource of the form j necessary for the production of one unit of production of the type i. Then, the cost of material resources for the production of products of the form i, taking into account the solution of problems (2)–(7) is calculated as follows:

$$\sum_{j=1}^{M} \alpha_j \sum_{i=1}^{N} l_{ij} \left(\int_O^T q_{ini}(t) dt \right)$$

Here, α_j is the cost of a unit of material resource of the form j.

Next, assume that the costs of material resources associated with the implementation of the production program maximising marginal income (2) should not be higher than the value of V, which sets the amount of funds allocated for the purchase of material production resources. Therefore, the following equation must be fulfilled:

$$\sum_{j=1}^{M} \alpha_j \sum_{i=1}^{N} l_{ij} \left(\int_O^T q_{ini}(t) dt \right) \leq V \tag{26}$$

As V can be taken the expenditure on material resources in the production of products in the volume of Zak_i.

Let us divide both sides of equation (26) by a positive value of V and denote:

$$y_i = \frac{\left(\int_O^T q_{ini}(t) dt \right) \sum_{j=1}^{M} \alpha_j l_{ij}}{V} \tag{27}$$

It is obvious that y_i is the share of funds spent on the purchase of material resources in a situation when the quantity of products of the type i is produced in the volume of: $\int_O^T q_{ini}(t) dt.$

In this case, the volatility of the revenues of the production program given as $\int_O^T q_{ini}(t) dt; \, (i = 1, 2, \ldots, N)$ is defined as follows:

$$\sum_{i=1}^{N} \sigma_i^2 y_i^2 + 2 \sum_{i=1}^{N} \sum_{j>i} \text{cov}_{ij} \, y_i y_j \tag{28}$$

Here, σ_i^2 is the variance of marginal income for products of the type i, cov_{ij} is the covariance of the marginal income of products of type i and products of type j (i = 1, 2, …, N, j = 1, 2, …, M).

Constraints on the volume of purchases of material resources in this case taking into account [equation (27)] will be as follows:

$$\sum_{i=1}^{N} y_i \leq 1;\ y_i \geq 0 \tag{29}$$

Equation (28) can be used to assess the risk of profitability of the production program, which can be used as an additional criterion for evaluating its effectiveness. Thus, in a situation where the marginal income is β_i (i = 1, 2, …, N), it is a random value with a given distribution law. The task of optimising the production program with regard to the constraints on the risk of its profitability can be formulated as follows:

$$\sum_{i=1}^{N} \overline{\beta}_i \int_O^T K_1^{ini} q_{ini}(t)dt \to \max \tag{30}$$

Here, $\overline{\beta}_i$ is the mathematical expectation of the marginal income per unit of output of the type i (i = 1, 2, …, N).

$$\begin{gathered} V_{ijk} + \int_o^t K_1^{ij-1} q_{ij-1k}(t')dt' + \sum_{Opg \in R_{ij}} \int_o^t K_2^{pg} q_{pg}(t')dt' \geq \int_o^t q_{ijk}(t')dt'; \\ i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n_i;\ k = 1, 2, \ldots, m,\ \forall t \in (0, T) \end{gathered} \tag{31}$$

$$\sum_{i=1}^{N} y_i \leq 1,\ y_i \geq 0 \tag{32}$$

$$y_i = \frac{\int_O^T q_{ini}(t)dt \sum_{j=1}^{M} \alpha_j l_{ij}}{V} \tag{33}$$

Here, y_i is the share of financial resources spent on the purchase of material resources in the production of products in volume $\int_O^T q_{ini}(t)dt$ ($i = 1, 2, \ldots, N$).

$$\sum_{i=1}^{N} \sigma_i^2 y_i^2 + 2\sum_{i=1}^{N} \sum_{j=1, j>i}^{N} \mathrm{cov}_{ij}\ y_i y_j \leq R_g \tag{34}$$

Here, R_g is the acceptable level of volatility (risk level) of the profitability of the production program.

$$\sum_{i=1}^{N} \sum_{j=1}^{n_i} \frac{q_{ijk}(t)}{q_{ijk}^o(t)} \leq C_l,\ l = 1, 2, \ldots, m \tag{35}$$

$$\int_O^T K_1^{ini} q_{ini}(t)dt \geq Zak_i\ \ i = 1, 2, \ldots, N \tag{36}$$

$$\int_O^T K_1^{ini} q_{ini}(t)dt \geq Pt_i\ \ i = 1, 2, \ldots, N \tag{37}$$

$$q_{ij}(t) \geq 0,\ i = 1, 2, \ldots, N;\ j = 1, 2, \ldots, n_i;\ \forall t \in (O, T). \tag{38}$$

Consider a situation where the demand for manufactured products is also given as a random variable, i.e.,

$$Pt_i \begin{array}{l} \nearrow Pt_i^1 - p_1 \\ \quad * \\ \searrow Pt_i^m - p_m \end{array} \qquad P_j \geq 0;\ j = 1, 2, \ldots, m$$

Here, Pt_i is the volume of demand for products of the type i, which is realised with probability P_j ($j = 1, 2, \ldots, m$, $i = 1, 2, \ldots, N$).

The mathematical expectation of demand in this case is estimated as:

$$\overline{P}t_i = \sum_{j=1}^{m} Pt_i^j P_j$$

The value $\overline{P}t_i$ can be used in models (2)–(7) as a demand constraint, but depending on the actual value of the random variable Pt_i, this demand may be more than $\overline{P}t_i$, as well as less than $\overline{P}t_i$, therefore it appears as a risk overproduction, and the risk of loss of profits due to the fact that products of type i turned out to be released in a smaller volume than the existing demand.

Consider how these risks can be assessed. The risk of loss of profits is estimated as the mathematical expectation of a decrease in profits due to the fact that the volume of output was lower than real demand.

The quantitative risk assessment of profit lost $P_{l.p.}$ can be determined by the following equation:

$$P_{l.p.} = \sum\nolimits_{i=1}^{N} \overline{\beta}_\iota \sum\nolimits_{j=1}^{m} \Delta_i^j P_j \tag{39}$$

In equation (39), $\overline{\beta}_\iota$ is the mathematical expectation of marginal income with the release of one unit of production of the type i, and Δ_i^j value is determined by the following equation:

$$\Delta_i^j = \begin{cases} 0, & \text{if } Pt_i^j - x_i \leq 0 \\ Pt_i^j - x_i, & \text{if } Pt_i^j > 0 \end{cases} \qquad j = 1, 2, \ldots, m;\ i = 1, 2, \ldots, N$$

Here, x_i is the volume of output of the type i, $x_i = \int_O^T K_1^{ini} q_{ini}(t) dt.$

The risk of overproduction can be quantified as the mathematical expectation of losses due to the fact that the volume of final products turned out to be greater than the actual demand for these products.

The equation for assessing this risk P_o is as follows:

$$P_o = \sum\nolimits_{i=1}^{N} b_i \sum\nolimits_{j=1}^{m} \theta_i^j P_j \tag{40}$$

In equation (40), the following notation is used: b_i – variable costs of the production of a unit of production of the type i and θ_i^j – is determined by the following equation:

$$\theta_i^j = \begin{cases} 0, & \text{if } x_i - Pt_i^j \le 0 \\ x_i - Pt_i^j, & \text{if } x_i - Pt_i^j > 0 \end{cases} \quad j = 1, 2, \ldots, m; i = 1, 2, \ldots, N$$

$$x_i = \int_0^T K_1^{ini} q_{ini}(t)dt.$$

Below, there are examples of quantitative assessment of the risk of loss of profits and the risk of overproduction.

Let two types of products be produced in volumes $x_1 = \int_O^T K_1^{1n_1} q_{1n_1}(t)dt = 20$ and $x_2 = \int_O^T K_1^{2n_2} q_{2n_2}(t)dt = 14.$

The demand for products of the first and second types is given as a random variable using Table 1.

Table 1 Demand for products of the first and second type

Probability	*Demand for products of the first type*	*Demand for products of the second type*
$P_1 = \frac{1}{2}$	20	16
$P_2 = \frac{1}{3}$	18	6
$P_3 = \frac{1}{6}$	24	24

Consider the mathematical expectation of demand for each type:

$$\overline{Pt_1} = \sum_{j=1}^{m} Pt_1^j P_j = 20 \cdot \frac{1}{2} + 18 \cdot \frac{1}{3} + 24 \cdot \frac{1}{6} = 10 + 6 + 4 = 20$$

$$\overline{Pt_2} = \sum_{j=1}^{m} Pt_2^j P_j = 16 \cdot \frac{1}{2} + 6 \cdot \frac{1}{3} + 24 \cdot \frac{1}{6} = 14$$

Thus, if the values $\overline{Pt_1}$ and $\overline{Pt_2}$ are used as the demand in the constraints of models (2)–(7), then the production program x = (20.14) is acceptable. Equations (39)–(40) can be used to determine the magnitude of the risk of lost profits and the risk of overproduction. Assume that $b_1 = 1{,}200, b_2 = 1{,}000, \overline{\beta_1} = 800, \overline{\beta_2} = 500.$

Taking into account these data and the data given in Table 1, the required calculations can be carried out:

$$P_{l.p.} = \sum_{i=1}^{N} \overline{\beta_i} \sum_{j=1}^{m} \Delta_i^j P_j = \frac{1}{6}(24 - 20) \cdot 800 + \frac{1}{2}(16 - 14) \cdot 500 + \frac{1}{6}(24 - 14) \cdot 500 = 1868$$

$$P_o = \sum_{i=1}^{N} b_i \sum_{j=1}^{m} \theta_i^j P_j = \frac{1}{3}(20 - 18) \cdot 1200 + \frac{1}{3}(14 - 6) \cdot 1000 = 3466.$$

3.4 Model of management of production and financial activities of the enterprise with regard to the defected products for each type of equipment

Consider the task of optimising the production and financial activities of an enterprise in conditions where coefficients reflecting the proportion of qualitative products, the share of recoverable defects and the share of unrecoverable defects for each type of equipment are defined. Thus, if K types of equipment are involved in the production process, then $K_1^l, K_2^l, K_3^l,$ respectively, are coefficients that determine the proportion of qualitative products, recoverable defected products and unrecoverable defects when processed on equipment of the type l (l = 1, 2, …, K). Assume that $K_1^l \geq 0, K_2^l \geq 0, K_3^l \geq 0,$ $\sum_{j=1}^{3} K_j^l = 1, \forall l = 1, 2, \ldots, K.$

Consider the model of production and financial planning in the absence of expansion of the production base and constraints on the company's working capital intended to the purchase of material resources in conditions if $K_1^l = 1, K_2^l = 0, K_3^l = 0,$ i.e., recoverable and unrecoverable manufacturing defect is missing. As the optimisation criterion, as before, the company's profit from the sale of the final product is chosen.

$$\sum_{i=1}^{N} a_i x_i - \sum_{i=1}^{N} b_i x_i - Z_{fixed} \to \max. \tag{41}$$

Here, a_i is the selling price of a unit of the product of the type i, b_i – variable costs when selling products i, Z_{fixed} – fixed costs, and x_i is the volume of output of the type i (i = 1, 2, …, N).

$$\sum_{j=1}^{M} W_j \sum_{i=1}^{N} l_{ij} x_i \leq V \tag{42}$$

Here, V is the volume of working capital directed to the purchase of material resources of production, and M – the number of types of material resources.

$$\sum_{i=1}^{N} t_{il} x_i \leq \tau_l K_l, l = 1, 2, \ldots, K \tag{43}$$

Here, t_{il} is the load time of equipment of the type l with the release of one unit of production of the type i, τ_l is the time during which equipment of the type l participates in the production process if the planning interval is set (O, T), and K_l is the number of units of equipment of the type l (l = 1, 2, …, K).

$$x_i \leq Pt_i, i = 1, 2, \ldots, N \tag{44}$$

$$x_i \geq Zak_i, i = 1, 2, \ldots, N \tag{45}$$

$$x_i \in Z^+ \tag{46}$$

Here, Z^+ is the set of non-negative integers. Under the conditions when $K_1^l = 1\,(l = 1, 2, \ldots, K);$ the loading time of equipment of the type l at the output of products of the type i in the volume x_i, as follows from equation (43), is equal to $t_{il}x_i$. In the situation if $K_1^l < 1,$ this time will be $(t_{il} x_i) / K_1^l, (l = 1, 2, \ldots, K).$

In other words, we can assume that the complexity of processing on equipment of the type l will not be t_{il}, but the value $\widetilde{t_{il}} = \frac{t_{il}}{K_1^l}, l = 1, 2, \ldots, K.$

Thus, under the conditions if $K_1^l < 1$ in models (41)–(46), constraint (43) changes to the constraint:

$$\sum_{i=1}^{N} \widetilde{t_{il}} x_i \leq \tau_l K_l, l = 1, 2, \ldots, K \tag{43.1}$$

Next, consider how recoverable and unrecoverable defects affect the constraints of models (41)–(46). It is obvious that the coefficient of unrecoverable defects $K_3^l > 0$ increases the consumption rate of material resources, therefore if $K_2^l = 0,$ then the volume of material resources of the type j in the production of a unit of type i increases from l_{ij} to $\widetilde{l_{iJ}} = \frac{l_{ij}}{1 - K_3^l}.$

Therefore, constraint (42) is modified as follows:

$$\sum_{j=1}^{M} W_j \sum_{i=1}^{N} \widehat{l_{ij}} x_i \leq V \tag{42.1}$$

Here, $\widehat{l_{ij}} = \frac{l_{ij}}{\prod_{q \in \theta_i} \left(1 - K_3^q\right)},$ and the set θ_i is the set of types of equipment that are used in the production of products of the type i.

Thus, the model of managing the production and financial activities of an enterprise in the situation of accounting for defects is given by equations (41), (42.1), (43.1) and (44)–(46).

3.5 *The analysis of stability in models of management of production and financial activities of the enterprise*

Consider a set of integer solutions among all the acceptable solutions in the optimisation models (2)–(7). Let us define this set by $\underline{\overline{X}} = \{x^1, x^2, \ldots, x^F\}$. It is assumed that $x_i^j = \int_O^T q_{iN}^j(t)dt, (i = 1, 2, \ldots, N, j = 1, 2, \ldots G)$ is an integer. Due to constraints (3)–(7) of the models (2)–(7), the set of integer production programs is finite and let us $x^l \in \underline{\overline{X}}$ be optimum.

Let us consider the situation of marginal profit β_i change in the target function (1) under influence of accumulated inflation. It is assumed that the value of β_i with the level of accumulated inflation changes according to the law:

$$\beta_i(\xi) = \beta_i(0) + \varphi_i(\beta_i, \xi) \tag{47}$$

Here, $\beta_i(\xi)$ is the value of marginal profit β_i provided that accumulated inflation ξ (in shares), $\beta_i(0)$ is the initial value of marginal profit, $\varphi_i(\beta_i\xi)$ is a non-decreasing differentiable function with the respect to the ξ variable, dependent on the parameter β_i and $\varphi_i(\beta_i 0) = 0$, $(i = 1, 2, \ldots, N)$.

Let us see the value of target function (2) taking into the account relation (47) at the optimal, with $\xi = 0$, the solution of tasks (2)–(7) and the level of accumulated inflation ξ:

$$F^l(\xi) = \sum_{i=1}^{N} \left(\beta_i(0) + \varphi_i\left(\beta_i, \xi\right)\right) K_1^{ini} x_i^l \tag{48}$$

Here, $F^l(\xi)$ is the value of target function (2) at the optimal integer solution x_i^l under the condition of accumulated inflation ξ.

Evidently, the value of target function (2) in any other production program x^j ($j = 1, 2, \ldots, G$) at the level of accumulated inflation j can be represented as follows:

$$F^j(\xi) = \sum_{i=1}^{N} \left(\beta_i(0) + \varphi_i\left(\beta_i, \xi\right)\right) K_1^{ini} x_i^l,\ j = 1, 2, \ldots, G \tag{49}$$

Let us consider the question whether the optimal solution x^l of tasks (2)–(7) stays the same or not when the level of accumulated inflation ξ changes.

Definition 1: Integer solution x^l of tasks (2)–(7) will be called stable if there is such a level of accumulated inflation $\tilde{\xi}$ that for all $0 \le \xi \le \tilde{\xi}$, the following equation is true:

$$F^l(\xi) \ge F^j(\xi) \quad \forall j = 1, 2, \ldots, G;\ j \ne l$$

Definition 2: Integer solution x^l of tasks (2)–(7) will be called absolutely stable if for all values of accumulated inflation $\xi \in (0, \infty)$, the following relation is true:

$$F^l(\xi) \ge F^j(\xi) \quad \forall j = 1, 2, \ldots, G;\ j \ne l$$

Evidently, within the framework of the formulated definitions a sufficient condition for the stability of the solution x_l of tasks (2)–(7) is uniqueness of the optimal solution. Indeed, if the optimal solution is unique, a positive value can be determined:

$$\Delta = \min\left(F^l(0) - F^j(0)\right),\ j = 1, 2, \ldots, G;\ j \ne l.$$

By the virtue of the continuity and differentiability $\varphi_i(\beta_i, \xi)$, one can choose such ξ^* that $\varphi_i(\beta_i \xi^*) < \frac{\Delta}{N};\ i = 1, 2, \ldots, N;$ this implies $F^j(\xi) < F^l(0)$; for all $0 \le \xi \le \xi^*, j = 1, 2, \ldots, G$; $j \ne 1$; therefore, $F^j(\xi) < F^l(\xi)$ for $0 \le \xi \le \xi^*$, implying that the solution x^l stability is sufficient.

As it is easy to see, a sufficient condition for the absolute stability of the solution x^l is the fulfilment of the equation:

$$\left(F^l(\xi)\right)' > \left(F^j(\xi)\right)',\ \forall \xi \in (0, \infty);\ \forall j = 1, 2, \ldots, G;\ j \ne l \tag{50}$$

Here, $(F^j(\xi))'$ ($j = 1, 2, \ldots, G$) are derivatives of the function $F^j(\xi)$ with respect to the level of accumulated inflation ξ.

After carrying out all the necessary transformations condition (50) can be brought to the following form:

$$\left(F^j(\xi)\right)' = \sum_{i=1}^{N} \left(\varphi_i\left(\beta_i, \xi\right)\right)' K_1^{ini} x_i^j,\ j = 1, 2, \ldots, G;\ j \ne l$$

$$\left(F^{l}(\xi)\right)' = \sum_{i=1}^{N} \left(\varphi_i\left(\beta_i, \xi\right)\right)' K_1^{ini} x_i^{l}$$

$$\sum\nolimits_{i=1}^{N} \left(\varphi_i\left(\beta_i, \xi\right)\right)' K_1^{ini} x_i^{l} > \sum\nolimits_{i=1}^{N} \left(\varphi_i\left(\beta_i, \xi\right)\right)' K_1^{ini} x_i^{j} \tag{50.1}$$

Here, $(\varphi_i(\beta_i, \xi))'$ are derivatives of the function $(\varphi_i(\beta_i, \xi))$ with respect to ξ $(i = 1, 2, \ldots, N)$.

In the situation where x^l is stable, there is a question how large might be accumulated inflation ξ, at which the optimal production program x^l will stay. Apparently, to answer this question, $G-1$ equation must be solved:

$$F^{l}(\xi) = F^{j}(\xi),\ j = 1, 2, \ldots, G;\ j \neq l \tag{51}$$

Let us denote the solutions of these equations through $\xi_1, \ldots, \xi_{\tilde{G}}$. It is obvious that the interval of the change ξ, with the change of inflation, on which the optimal production program x^l will be saved, is defined as $(0, \xi_{\min})$, where $\xi_{\min} = \min \xi_j$ $(j = 1, 2, \ldots, G, j \neq 1)$.

Let us consider the situation of linear growth β_i from the level of accumulated inflation ξ, i.e., it is assumed:

$$\beta_i \xi = \beta_i(0) + n_i \beta_i(0)\xi \tag{52}$$

In this case, equation (51) can be rewritten in the following way:

$$\sum\nolimits_{i=1}^{N} \left(\beta_i(0) + n_i \beta_i(0)\xi\right) K_1^{ini} x_i^{l} = \sum\nolimits_{i=1}^{N} \left(\beta_i(0) + n_i \beta_i(0)\xi\right) K_1^{ini} x_i^{j}, \quad j = 1, 2, \ldots, G;\ j \neq l \tag{53}$$

Considering the linearity of equation (53), we can say that each of them has at most one solution. The same can be said about any equation of the type:

$$F^{j}(\xi) = F^{P}(\xi),\ P = 1, 2, \ldots, G,\ j = 1, 2, \ldots, G;\ P \neq j \tag{54}$$

with the execution of the ratio [equation (52)].

Therefore, the following statements can be formulated.

With a linear change of marginal income from accumulated inflation ξ [equation (52)] and in the situation where $\xi \in (0, \infty)$, there is a finite number of points $\xi_1, \xi_2, \ldots, \xi_P$, where the transition to the new optimal production program from the set $\underline{\overline{X}}$ takes place. In the situation, if the growth of marginal income from accumulated inflation is nonlinear, then already in a situation where only two production programs are included into the set $\underline{\overline{X}}$ $(\underline{\overline{X}} = \{x^1, x^2\})$, the number of points of transformation from optimal program x^1 to program x^2 and from program x^2 to program x^1 might be infinite due to the possibility of infinite number of solutions for nonlinear equation (54).

3.6 The model of the choice of optimal production program based on the various equipment performance

Let us consider the problem of choosing the optimal production program in a situation when the material flow processing at production operations is done with the use of

K units of equipment with the performance α_{ij}^{l}, where α_{ij}^{l}, is performance of the equipment l on operation O_{ij} ($l = 1, 2, \ldots, K$, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, n_i$). As before, let us set the task of choosing the optimal production program according to the marginal income maximisation criteria in the context of constraints on performance of the equipment involved in the production process.

$$\sum_{i=1}^{N} \int_{O}^{T} K_1^{ini} q_{ini}(t)dt \to \max \tag{55}$$

$$V_{ijk}(0) + \int_{o}^{t} K_1^{ij-1} q_{ij-1k}(t')dt + \sum_{Opg \in R_{ij}} \int_{o}^{t} K_2^{pg} q_{pgk}(t')dt \geq \int_{o}^{t} q_{ijk}(t')dt' \tag{56}$$
$$\forall t \in (O, T);\, i = 1, 2, \ldots, N;\; j = 1, 2, \ldots, n_i;\, K = 1, 2, \ldots, M$$

$$q_{ij}(t) \leq \sum_{l=1}^{K} \alpha_{ij}^{l} \theta_{ij}^{l}(t) \quad \forall t \in (O, T);\, i = 1, 2, \ldots, N;\; j = 1, 2, \ldots, n_i$$
$$\theta_{ij}^{l}(t) = \begin{cases} 1 - \text{if in ther moment } t \text{ the equipment } l \text{ is doing the material} \\ \text{processing at operation } O_{ij} \\ 0 - \text{otherwise} \end{cases} \tag{57}$$

$$\sum_{i=1}^{N} \sum_{j=1}^{n_i} \theta_{ij}^{l}(t) \leq 1,\, \forall t \in (O, T),\, l = 1, 2, \ldots K \tag{58}$$

Constraint (58) tells that in every moment of time t, each unit of equipment is used only at one operation.

$$\int_{O}^{T} K_1^{ini} q_{ini}(t)dt \geq Zak_i,\, i = 1, 2, \ldots, N \tag{59}$$

$$\int_{O}^{T} K_1^{ini} q_{ini}(t)dt \geq Pt_i,\, i = 1, 2, \ldots, N \tag{60}$$

$$q_{ij}(t) \geq 0,\, i = 1, 2, \ldots, N;\; j = 1, 2, \ldots, n_i \tag{61}$$

Problems (55)–(61) as well as problems (2)–(7) is the problem of optimal control, which solution is the functions $q_{ij}(t)$ and $O_{ij}^{l}(t)$.

Considering the difficulty of getting the optimal solution to the problems (55)–(61) in the general case, it can be reduced to the problem of linear programming with the use of function $q_{ij}(t)$ approximation with piecewise constant functions. In this situation, the problem of marginal income optimisation can be formulated as following:

$$\sum_{\tau=1}^{T} \sum_{i=1}^{N} \sum_{l=1}^{K} \beta_i^{\tau} x_{in_i}^{\tau l} \alpha_{in_i}^{l} K_1^{ini} \to \max \tag{62}$$

Here, it is assumed that the period (O, T) is divided into T days, β_i^{τ} is a marginal income from product i sales during the day number τ, $x_{in_i}^{\tau l}$ is a fraction of the day τ, which the equipment with the number l uses on operation O_{ini}.

$$V_{ij\tau}(0)+\sum_{\tau=1}^{q}\sum_{l=1}^{K}K_1^{ij-1}x_{ij-1}^{\tau l}\alpha_{ij-1}^{l}+\sum_{\tau=1}^{q}\sum_{l=1}^{K}\sum_{Opg\in R_{ij}}K_2^{pg}x_{pg}^{\tau l}\alpha_{pg}^{l} \geq \sum_{\tau=1}^{q}\sum_{l=1}^{K}x_{ij}^{\tau l}\alpha_{ij}^{l} \quad i=1,2,\ldots,N;\ j=1,2,\ldots,n_i;\ \tau=1,2,\ldots,M;\ q=1,2,\ldots,T \tag{63}$$

$x_{ij}^{\tau l}$ fraction of the day, which the equipment l uses at operation O_{ij} during the day τ.

$$\sum_{i=1}^{N}\sum_{j=1}^{n_i}x_{ij}^{\tau l}\leq 1 \quad i=1,2,\ldots,N;\ j=1,2,\ldots,n_i;\ \forall\tau=1,2,\ldots,T \tag{64}$$

$$\sum_{\tau=1}^{T}\sum_{l=1}^{K}x_{in_i}^{\tau l}\alpha_{in_i}^{l}\geq Z_{aki} \quad i=1,2,\ldots,N \tag{65}$$

$$\sum_{\tau=1}^{T}\sum_{l=1}^{K}x_{in_i}^{\tau l}\alpha_{in_i}^{l}\geq Pt_i \quad i=1,2,\ldots,N \tag{66}$$

$$x_{ij}^{\tau l}\geq 0,\ x_{ij}^{\tau l}\leq 1 \tag{67}$$

Here, $x_{ij}^{\tau l}$ is the fraction of the day τ, which the equipment l uses at operation O_{ij}.

Discretisation of models (55)–(61) and the finding of the solution on the set of piecewise constant functions allowed reducing the problem of optimal control to the problem of linear programming.

Generally, the results of this work are the development of: the complex of optimal control models, the tool to convert an optimal control problem to linear programming model, the evaluation mechanism of the stability of solutions under changes in the prices of end products, risk assessment methods of the obtained solution.

4 The example of the production and financial activity calculation

Let us consider the simplified business process diagram of the small enterprise for assembling system blocks of personal computers of three types: business, gaming and home and office. The given enterprise purchases components for system blocks from a wholesale supplier and stores them at their own warehouse. Employees of the company with the use of four types of equipment (mobile terminal for picking, assembly stand, testing stand and packing stand) with four types of operations (picking, assembling, testing and packing) prepare the finished product for shipment. The graphic scheme of production business-process is shown in Figure 2.

Let us solve the problem of the annual profit optimisation of this production company using the following initial data (Tables 2, 3, 4 and 5). It is worth mentioning that the original stochastic data was converted to deterministic form for better understanding (see Section 3.3).

Table 2 Annual demand in deterministic form

Finished products	*Price, RUB thousands*	*Orders, pcs.*
Computer for business	110	360
Gaming computer	140	460
Computer for home and office	30	550

Table 3 Constraints on production resources during the year

Resources	*Availability*
Worker	2 people
Mobile terminal	1 pc.
Assembly stand	1 pc.
Testing stand	1 pc.
Packing stand	1 pc

Note: Total annual payroll is 540,000 rubles.

Table 4 Performance for all types of finished products

Process	*Resource 1*	*Resource 2*	*Performance, sets per day*	*Performance, sets per year*
Picking	Mobile terminal	Worker	32	7,680
Assembling	Assembly stand	Worker	14	3,360
Testing	Testing stand	Worker	28	6,720
Packing	Packing stand	Worker	32	7,680

It is worth mentioning that the performance values already takes into account possible recoverable and unrecoverable defects based on statistical data.

Table 5 Price of components

Components	*Price, RUB thousand*	*Initial stock, pcs.*
System unit (business)	35	5
Memory (business/gaming)	27	5
Processor (business/gaming)	30	5
System unit (gaming)	35	5
Graphic map (gaming)	30	5
System unit (home/office)	9	5
Memory (home/office)	4	5
Processor (home/office)	7	5

In order to solve this problem, let us use the specialised software for supply chain planning – River Logic Enterprise Optimizer (River Logic, 2008) (see Figure 3).

The optimisation results are shown in Table 6.

Workload of the resources is shown in Table 7.

Let us make a task more complicated and introduce two additional annual periods. For these periods, we set the inflation 5% per year. Let us suppose that the enterprise is starting to grow rapidly. Aggregated demand increases by 100% each year. The costs also grow by 3% each year due to currency fluctuations and inflation. It is necessary to determine the necessary volume of investment into the production equipment and in new employee attraction with the target function to optimise the net present value (discount rate is 5%). Primary costs on equipment and employee are shown in Table 8.

Table 6 Financial results of single period model optimisation

	Computer for business	*Gaming computer*	*Computer for home and office*	*Total*
	Pcs.	*Pcs.*	*Pcs.*	*Pcs.*
Execution	360	460	550	1,370
	RUB thousand	*RUB thousand*	*RUB thousand*	*RUB thousand*
Revenue	39,600	64,400	16,500	120,500
Purchase	32,820	55,635	10,900	99,355
Stock	300	485	100	885
Total materials	33,120	56,120	11,000	100,240
Total payroll	284	363	434	1,080
Production costs	33,404	56,483	11,434	101,320
Gross profit	6,196	7,917	5,066	19,180

Table 7 Workload of the resources in single period model

Process	*Number of processed sets, pcs.*	*Resource 1*	*Workload*	*Resource 2*	*Workload*
Packing	1370	Worker	18%	Packing stand	36%
Testing	1370		20%	Testing stand	41%
Assembly	1370		41%	Assembly stand	82%
Picking	1370		18%	Mobile terminal	36%
Total			97%		

Table 8 Primary investment on the unit of resource

Resource	*Number of primary investment, RUB thousand*
Assembly stand	10
Packing stand	8
Testing stand	9
Mobile terminal	40
Worker	6.5

The results of optimisation are shown in Table 9.

Table 9 Resource demand in multi-period model

Resource	*Number of needed resources*			*Needed investment, RUB thousand*	
	1st year	*2nd year*	*4th year*	*2nd year*	*3d year*
Assembly stand	1	2	3	10	10
Packing stand	1	1	2		8
Testing stand	1	1	2		9
Worker	2	4	6	13	13
Mobile terminal	1	1	2		40
Total				23	80

Table 10 Financial results of multi-period model optimisation

Item	*Computer for business*			*Gaming computer*			*Home and office computer*			*Total*		
Period	*1*	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*	*1*	*2*	*3*
		Pcs.			*Pcs.*			*Pcs.*			*Pcs.*	
Execution	550	1,100	1,650	460	920	1,380	360	720	1,080	1,370	2,740	4,110
		RUB thousand			*RUB thousand*			*RUB thousand*			*RUB thousand*	
Revenue	39,600	83,160	130,977	64,400	135,240	213,003	16,500	34,650	54,574	120,500	253,050	398,554
Purchase	32,820	68,227	105,345	55,635	115,607	178,523	10,900	22,660	34,994	99,355	212,226	313,130
Stock	300			485			100			885	0	0
Materials in total	33,120	68,227	105,345	56,120	115,607	178,523	11,000	22,660	34,994	100,240	212,226	313,130
Total payroll	284	585	903	363	747	1,154	434	893	1,380	1,080	2,225	3,437
Production Cost	33,404	68,812	106,248	56,483	116,354	179,677	11,434	23,553	36,374	101,320	214,451	316,567

Financial results can be seen in Table 10.

Figure 2 Graphic scheme of production business-process

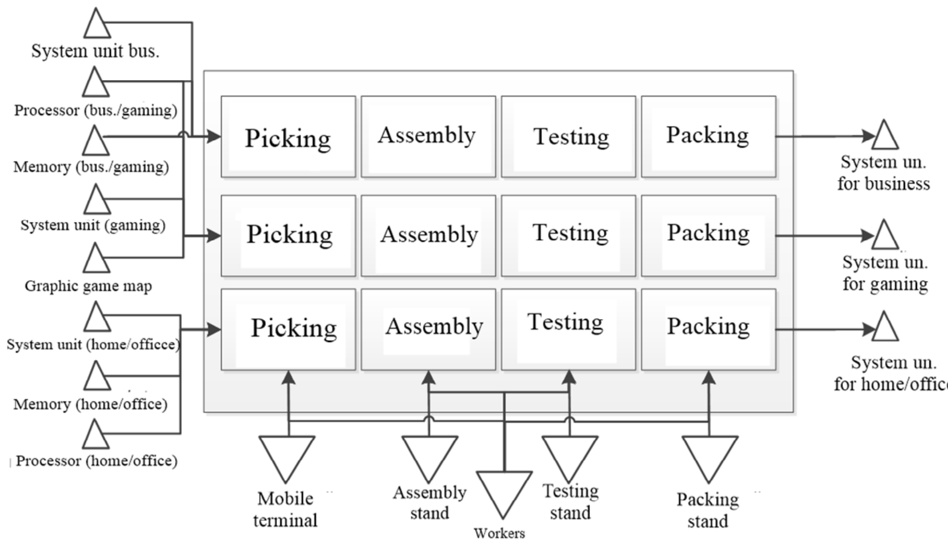
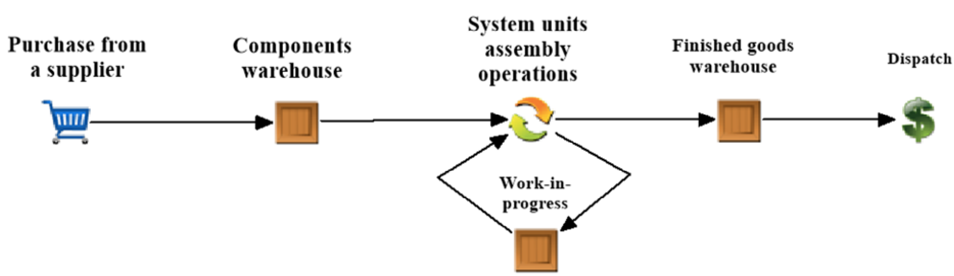


Figure 3 Enterprise material flow scheme in enterprise optimiser (see online version for colours)



Accordingly, taking into account the investments net present value is (in RUB thousand): 1 year – 18,267, 2 years – 34,989, 3 years – 70,734; in total 123,990. The data presented speaks in favour of adopting investment program since a significant economic effect is achieved with the relatively small investments.

5 Conclusions

The paper presents unique dynamic models for managing constrained resources, taking into account uncertainty and risk in industrial logistics, which will improve the efficiency of industrial logistics systems both in terms of capacity utilisation and in terms of using material resources of production. The considered dynamic models consider recoverable and unrecoverable defects. The developed methodological tools for managing constrained resources of industrial enterprises confirmed in practice its effectiveness which was shown particularly in evaluating the effectiveness of the investment program of the manufacturer of computers.

The results of this work are the development of: the complex of optimal control models taking into account uncertainty and risk, the tool to convert an optimal control problem to linear programming model, the evaluation mechanism of the stability of solutions under changes in the prices of end products, risk assessment methods of the obtained solution.

In the future, it is planned to develop this mathematical apparatus, both in terms of the criteria used, and in terms of additional constraints when choosing the optimal production program.

Acknowledgements

The paper is written as a part of the studies conducted under the grant of RFBR, Project No. 16-06-0043a.

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