Correlation Functions of the Noise Modulation Function under the Influence of Stationary and Slow Multiplicative Noise

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***Abstract*—An approach for determining the correlation functions of the noise modulation function under the influence of modulating (or more often multiplicative) noise is described. Both stationary and slow noise is considered as influencing multiplicative noise. Correlation functions of noise modulation and fluctuations of the latter can be found by the four-dimensional characteristic function of phase and amplitude distortions. We have derived expressions for the above-mentioned correlation functions affected by static multiplicative noise for the normal law of variation of amplitude and phase distortions in the case of their uncorrelation and cross-correlation. The correlation functions of noise modulation and noise modulation function fluctuations under the influence of slow modulating noise were determined for the case when the phase and distortion amplitude variations are independent. The expressions of the correlation functions of noise modulation and its fluctuations for correlated and normal distribution of phase and amplitude distortions under the influence of slow modulating noise are obtained.**

Keywords*—modulating (multiplicative) noise, noise modulation function, noise modulation function fluctuations, correlation function, amplitude distortion, phase distortion, characteristic function, slow multiplicative noise

I. Introduction

As is known, multiplicative (modulating) noise (MN) strongly influences on the reception of useful signals, as well as the performance characteristics of radio systems and devices [1-5, etc.]. The characteristics of the quality of operation of a particular radio system are determined by its purpose and are often specific to a particular type of system. In most cases, these characteristics are determined by some primary characteristics that describe the quality of signal reception in the presence of noise, such as the accuracy of determining signal information parameters, the resolution of a certain parameter, the probability of correct detection or extraction of the signal [6-11, etc.].

Note that the noise modulation function (NMF) can fully characterize MN. Correlation function (CF) and energy spectrum are the main characteristics of NMF.

The purpose of the work is to determine the CF of both the NMF and the fluctuations of the named function under the influence of stationary and slow MN during the mutual connection of changes in amplitude and phase of distortion, as well as when the latter represent a narrow-band random process.

In [12], the issue related to studying the spectra of a harmonic oscillation modulated in amplitude and phase by deterministic and quasi-deterministic functions and by stationary broadband fluctuation processes are analyzed in detail. Cases when phase and amplitude changes are interrelated, when they represent a narrow-band random process, as well as cases when phase changes are a non-stationary pulse-fluctuation process were considered much less often. Let us consider and analyze the CF and energy spectra of the NMF when the signal is influenced by MN. We will confine ourselves to considering the so-called slow MN, when the correlation time or the period of the NMF is longer than the duration of the coherently processed signal or a signal burst or of the same order with it.

As a rule, narrow-band signals whose spectrum width is much less than the carrier frequency are used in radio systems. Such signals, formed by modulation of a harmonic oscillation in amplitude and in phase (or frequency), write down as follows

$$u(t) = U(t)\cos\left[\omega_c t + \Phi(t) + \varphi_0\right],$$

this expression includes: the envelope of the signal $U(t)$, which is determined by the law of amplitude modulation of the signal, ω_c is the frequency of the carrier signal, and φ_0 is its initial phase. In turn, $\Phi(t)$ is used, which determines the law of phase modulation of the signal. If the frequency modulation law $\Omega(t)$ is used, the expression $\Phi(t) = \int \Omega(t)\,dt$, is applicable.

As is known, as a result of MN, amplitude and phase distortions of the signal appear. Let's write the expression for such a signal under the influence of MN

$$u_M(t) = \eta(t)U(t)\cos\left[\omega_c t + \Phi(t) + \varphi_0 + \varphi(t)\right]. \quad (1)$$

Here are presented $\varphi(t)$ phase distortions (changes in the phase of a useful signal) under the influence of MN, a dimensionless factor $\eta(t) \geq 0$ characterizes amplitude distortions of the signal, that is, changes in its envelope under the influence of MN.

Let's set the condition that the mentioned distortions have the character of parasitic modulation. In other words, the width of the function spectrum $\varphi(t)$ and $\eta(t)$ is significantly less than the frequency ω_c.

The expression $U_M(t)=\eta(t)U(t)$. defines the envelope of the signal $u_M(t)$.

Expression (1) can be represented as [13]:

$$u_M(t)=\mathrm{Re}\left[\dot{U}_M(t)\exp\{j(\omega_c t+\varphi_0)\}\right].$$

The latter expression includes a complex envelope $\dot{U}_M(t)=\dot{U}(t)\eta(t)\exp\{i\varphi(t)\}=\dot{U}(t)\dot{M}(t)$ of a signal $u_M(t)$ under the influence of MN, and a complex envelope of an undistorted signal is also used. Parasitic signal modulation, as noted above, will be fully described by NMF $\dot{M}(t)=\eta(t)\exp\{i\varphi(t)\}$.

The complex value in the given expression and further is indicated by a dot above one or another symbol.

Thus, as a result of the influence of MN, the complex envelope of the useful signal, which contains the transferred information, changes, and it leads to distortion of this information.

Let us consider and analyze CF of the NMF.

II. Correlation functions of noise modulation function with stationary fluctuation multiplicative noise

If the changes in phase and amplitude of the signal caused by MN are stationary processes, while in the case of the interdependence of the phase and amplitude, they are stationary related, then the NMF $\dot{M}(t)$ is stationary.

In this case, the NMF can be written as [14]

$$\dot{M}(t)=\bar{\dot{M}}+\dot{V}_0(t), \tag{2}$$

Here: the time-independent mathematical expectation of the FPM is given by $\bar{\dot{M}}=m_1\{\dot{M}(t)\}$, and the existing fluctuations of the FPM are described by $\dot{V}_0(t)$.

Averaging over a set in the given expression and further is indicated by a line above one or another symbol.

The CF of NMF $\dot{B}_M(\tau)$ and the CF of fluctuations of the NMF $\dot{B}_V(\tau)$ can be determined by the characteristic function of phase and amplitude distortions, where τ is the time of the variable delay of the envelope $\dot{U}_M(t)$.

To determine changes in phase and amplitude at moments in time $t_1=t$ and $t_2=t-\tau$, we use a four-dimensional characteristic function (FDCF) [13]:

$$\theta_4^{\eta\varphi}(x_1,x_2,x_3,x_4)=m_1\{\exp\{j(x_1\eta_1+x_2\eta_2+x_3\varphi_1+x_4\varphi_2)\}\},$$

where η_1 and φ_1 are the values $\eta(t)$ and $\varphi(t)$ at $t_1=t$, η_2 and φ_2 at $t_2=t-\tau$.

If the function $\theta_4^{\eta\varphi}$ is used, the CF $\dot{B}_M(\tau)$ can be written as

$$\begin{aligned}\dot{B}_M(\tau)&=m_1\{\dot{M}(t)M^*(t-\tau)\}=\\&=-\left[\frac{\partial^2}{\partial x_1\partial x_2}\theta_4^{\eta\varphi}(x_1,x_2,1,-1)\right]_{\substack{x_1=0\\x_2=0}}.\end{aligned} \tag{3}$$

The complex conjugate value in the given expression and further is indicated by a line above one or another symbol.

Taking into account (2), the CF of fluctuations is equal to

$$\dot{B}_V(\tau)=m_1\{\dot{V}_0(\mathrm{t})V_0^*(t-\tau)\}=\dot{B}_M(\tau)-\left|\bar{\dot{M}}\right|^2. \tag{4}$$

The mathematical expectation $\bar{\dot{M}}$ of the NMF can be expressed in terms of a two-dimensional characteristic function of change in phase and amplitude at coinciding time points $\theta_2^{\eta\varphi}(x_1,x_2)$. Thus,

$$\theta_2^{\eta\varphi}(x_1,x_2)=m_1\{\exp\{j(x_1\eta+x_2\varphi)\}\},$$

consequently,

$$\bar{\dot{M}}=m_1\{\eta\exp\{j\varphi\}\}=-j\left[\frac{\partial}{\partial x_1}\theta_2^{\eta\varphi}(x_1,1)\right]_{x_1=0}. \tag{5}$$

Taking (3)-(5) into account, the CF of fluctuations of the NMF is determined by the ratio

$$\begin{aligned}&\dot{B}_V(\tau)=\\&=\left[\frac{\partial^2}{\partial x_1\partial x_2}\theta_4^{\eta\varphi}(x_1,x_2,1,-1)\right]_{\substack{x_1=0\\x_2=0}}-\left|\left[\frac{\partial}{\partial x_1}\theta_2^{\eta\varphi}(x_1,1)\right]_{x_1=0}\right|^2.\end{aligned}$$

In many cases, a dimensionless multiplier $\eta(t)$ that determines amplitude distortions can be represented as

$$\eta(t)=\eta_0[1+\xi(t)]. \tag{6}$$

The expression (6) uses the mathematical expectation η_0 of a dimensionless factor $\eta(t)$, as well as a stationary random process with a zero mean $\xi(t)$, for which the following condition must be met: $[1+\xi(t)]\geq 0$. Thus, the CF of the NMF is determined as follows

$$\begin{aligned}\dot{B}_M(\tau)=\eta_0^2\Bigg\{&\theta_4^{\xi\varphi}(0,0,1,-1)-j\left[\frac{\partial}{\partial x_1}\theta_4^{\xi\varphi}(x_1,0,1,-1)\right]_{x_1=0}-\\&-j\left[\frac{\partial}{\partial x_2}\theta_4^{\xi\varphi}(0,x_2,1,-1)\right]_{x_2=0}-\\&-\left[\frac{\partial^2}{\partial x_1\partial x_2}\theta_4^{\xi\varphi}(x_1,x_2,1,-1)\right]_{\substack{x_1=0\\x_2=0}}\Bigg\},\end{aligned} \tag{7}$$

where $\theta_4^{\xi\varphi}(x_1,x_2,x_3,x_4)=m_1\{\exp\{j(x_1\xi_1+x_2\xi_2+x_3\varphi_1+x_4\varphi_2)\}\}$ is a FDCF $\xi(t)$ and $\varphi(t)$ at time points $t_1=t$ и $t_2=t-\tau$.

Let's use the expression to define a mathematical expectation of the NMF

$$\bar{\dot{M}}=\eta_0\theta_2^{\xi\varphi}-j\eta_0\left[\frac{\partial}{\partial x_1}\theta_2^{\xi\varphi}(x_1,1)\right]_{x_1=0}, \tag{8}$$

where $\theta_2^{\xi\varphi}(x_1,x_2)=m_1\{\exp(j(x_1\xi+x_2\varphi))\}$ is a two-dimensional characteristic function $\xi(t)$ and $\varphi(t)$ at coinciding moments of time.

Substituting (7) and (8) in (4), we obtain an expression for determining the CF of fluctuations in NMF:

$$\dot{B}_V(\tau) = \eta_0^2 \left\{ \theta_4^{\xi\varphi}(0,0,1,-1) - j\left[\frac{\partial}{\partial x_1}\theta_4^{\xi\varphi}(x_1,0,1,-1)\right]_{x_1=0} - \right.$$
$$-j\left[\frac{\partial}{\partial x_2}\theta_4^{\xi\varphi}(0,x_2,1,-1)\right]_{x_2=0} -$$
$$-\left[\frac{\partial^2}{\partial x_1 \partial x_2}\theta_4^{\xi\varphi}(x_1,x_2,1,-1)\right]_{\substack{x_1=0\\x_2=0}} - \tag{9}$$
$$\left. -\left|\theta_2^{\xi\varphi}(0,1) - j\left[\frac{\partial}{\partial x_1}\theta_2^{\xi\varphi}(x_1,1)\right]_{x_1=0}\right|^2 \right\}.$$

These formulas make it easy to determine $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$, if characteristic functions following the distribution law of amplitude and phase distortions are known. For example, if the distribution of phase $\varphi(t)$ and amplitude $\xi(t)$ distortions is close to normal, which is quite a common case, then the characteristic functions $\theta_4^{\xi\varphi}(x_1,x_2,x_3,x_4)$ and $\theta_2^{\xi\varphi}(x_1,x_2)$ can be represented as:

$$\theta_4^{\xi\varphi}(x_1,x_2,x_3,x_4) =$$
$$= \exp\{-0.5[\sigma_\xi^2 x_1^2 + \sigma_\xi^2 x_2^2 + \sigma_\varphi^2 x_3^2 + \sigma_\varphi^2 x_4^2 +$$
$$+2\sigma_\xi^2 r_\xi(\tau) x_1 x_2 + 2\sigma_\varphi^2 r_\varphi(\tau) x_3 x_4 + \tag{10}$$
$$+2\sigma_\xi \sigma_\varphi r_{\xi\varphi}(0) x_1 x_3 + 2\sigma_\xi \sigma_\varphi r_{\xi\varphi}(0) x_2 x_4 \times$$
$$\times 2\sigma_\xi \sigma_\varphi r_{\xi\varphi}(\tau) x_1 x_4 \, 2\sigma_\xi \sigma_\varphi r_{\xi\varphi}(-\tau) x_2 x_3]\},$$

$$\theta_2^{\xi\varphi}(x_1,x_2) =$$
$$= \exp\left\{-\frac{1}{2}\left[\sigma_\xi^2 x_1^2 + \sigma_\varphi^2 x_2^2 + 2\sigma_\xi \sigma_\varphi r_{\xi\varphi}(0) x_1 x_2\right]\right\}, \tag{11}$$

where σ_φ^2 and σ_ξ^2 are, respectively, the variances $\varphi(t)$ and $\xi(t)$, $r_\varphi(\tau)$ and $r_\xi(\tau)$ are their correlation coefficients, $r_{\xi\varphi}(\tau)$ is a cross-correlation coefficient $\varphi(t)$ and $\xi(t)$.

Substituting (10) and (11) into (7) and (9), we get

$$\dot{B}_M(\tau) = \eta_0^2 \{1 + \sigma_\xi^2 r_\xi(\tau) + j\sigma_\xi \sigma_\varphi [r_{\xi\varphi}(-\tau) - r_{\xi\varphi}(\tau)] +$$
$$+\sigma_\xi^2 \sigma_\varphi^2 [r_{\xi\varphi}(0) - r_{\xi\varphi}(\tau)] \times \tag{12}$$
$$\times [r_{\xi\varphi}(0) - r_{\xi\varphi}(-\tau)] \exp\{-\sigma_\varphi^2 [1 - r_\varphi(\tau)]\}\},$$

$$\dot{B}_V(\tau) = \eta_0^2 \{1 + \sigma_\xi^2 r_\xi(\tau) + j\sigma_\xi \sigma_\varphi [r_{\xi\varphi}(-\tau) - r_{\xi\varphi}(\tau)] +$$
$$+\sigma_\xi^2 \sigma_\varphi^2 [r_{\xi\varphi}(0) - r_{\xi\varphi}(\tau)][r_{\xi\varphi}(0) - r_{\xi\varphi}(-\tau)] \times \tag{13}$$
$$\times \exp\{-\sigma_\varphi^2 [1 - r_\varphi(\tau)]\} - \eta_0^2 \exp\{-\sigma_\varphi^2 [1 + \sigma_\varphi^2 \sigma_\xi^2 r_{\xi\varphi}^2(\tau)]\}.$$

If the amplitude and phase distortions are not correlated, then (12) and (13) are simplified:

$$\dot{B}_M(\tau) = \eta_0^2 \{1 + \sigma_\xi^2 r_\xi(\tau)\} \exp\{-\sigma_\varphi^2 [1 - r_\varphi(\tau)]\};$$
$$\dot{B}_V(\tau) = \tag{14}$$
$$= \eta_0^2 \{1 + \sigma_\xi^2 r_\xi(\tau)\} \exp\{-\sigma_\varphi^2 [1 - r_\varphi(\tau)]\} - \eta_0^2 \exp\{-\sigma_\varphi^2\}.$$

With pure phase distortion

$$\dot{B}_M(\tau) = \exp\{-\sigma_\varphi^2 [1 - r_\varphi(\tau)]\};$$
$$\dot{B}_V(\tau) = \exp\{-\sigma_\varphi^2\}[\exp\{\sigma_\varphi^2 r_\varphi(\tau)\} - 1].$$

Let's turn to the next important case of the so-called slow MN. For these noises, the condition is met $\tau_c >> T$, that is, the correlation time of the MN τ_c will be much longer than the signal with duration T on which these noise affects. Another feature of slow MNs under the above conditions is a rather slow change in the phase and amplitude of the processed signal under the influence of these noises.

III. Correlation Functions of NMF with Slow Multiplicative Noise

We assume that random processes $\varphi(t)$, $\eta(t)$ and $\xi(t)$ are root-mean-square differentiable. Then the CF $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ can be expanded into a McLaren series [15].

From physical considerations, it follows that when considering the influence MN on the signal with duration T, we will focus on the values of CF $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ at $|\tau| \le T$. If the correlation time τ_c of these functions is much longer than the signal duration T, then for $|\tau| \le T << \tau_c$ we can only take the first three terms of the series:

$$\dot{B}_M(\tau) \simeq \dot{B}_M(0) + \tau \dot{B}'_M(0) + 0,5r^2 \dot{B}''_M(0);$$
$$\dot{B}_V(\tau) \simeq \dot{B}_V(0) + \tau \dot{B}'_V(0) + 0.5r^2 \dot{B}''_V(0),$$

where $\dot{B}'_b(\tau) = \frac{d}{d\tau}\dot{B}_b(\tau)$, $\dot{B}''_b(\tau) = \frac{d^2}{d\tau^2}\dot{B}_b(\tau)$, $b = M, V$.

We will determine the values $\dot{B}_M(0)$, $\dot{B}'_M(0)$, $\dot{B}''_M(0)$, $\dot{B}_V(0)$, $\dot{B}'_V(0)$ and $\dot{B}''_V(0)$ for the case when the phase $\varphi(t)$ and amplitude $\xi(t)$ changes are independent.

By definition, the functions $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ can be written as

$$\dot{B}_M(\tau) = m_1\{\eta(t)\eta(t-\tau)\exp\{j[\varphi(t) - \varphi(t-\tau)]\}\};$$
$$\dot{B}_V(\tau) = \dot{B}_M(\tau) - |\bar{\dot{M}}|^2.$$

Thus

$$\dot{B}_M(0) = \overline{\eta^2} = \eta_0^2(1 + \sigma_\xi^2);$$
$$\dot{B}_V(0) = \overline{\eta^2} - |\bar{\dot{M}}|^2 = \eta_0^2(1 + \sigma_\xi^2) - |\bar{\dot{M}}|^2,$$

where $\eta^2 = m_1\{\eta^2(t)\}$, σ_ξ^2 is the variance $\xi(t)$.

Since the operations of differentiation and statistical averaging are interchangeable [15], then for the first derivative of the functions $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ we get

$$\dot{B}'_M(\tau) = \dot{B}'_V(\tau) =$$
$$= m_1\{\eta(t)\eta'(t-\tau)\exp\{j[\varphi(t) - \varphi(t-\tau)]\}\} - \tag{15}$$
$$-jm_1\{\eta(t)\eta(t-\tau)\varphi'(t-\tau)\exp\{j[\varphi(t) - \varphi(t-\tau)]\}\},$$

where

$$\eta'(t-\tau)=\frac{d}{d\tau}\eta(t-\tau);\quad \varphi'(t-\tau)=\frac{d}{d\tau}\varphi(t-\tau).$$

Then

$$\begin{aligned}\dot{B}'_{\mathrm{M}}(0)=\dot{B}'_{V}(0)=m_1\{\eta(t)\eta'(t)-jm_1\{\eta^2(t)\varphi'(t)\}\}=\\=-jm_1\{\{\eta^2(t)\varphi'(t)\}\},\end{aligned}\qquad(16)$$

where $\varphi'(t)=[\varphi'(t-\tau)]_{\tau=0}$; $\eta'(t)=[\eta'(t-\tau)]_{\tau=0}$.

Note, that the expression (16) takes into account, that $m_1\{\eta(t)\eta'(t)\}=0,$ when $\eta(t)$ is a stationary function.

If $\eta(t)$ and $\varphi(t)$ are independent, then $m_1\{\eta^2(t)\varphi'(t)\}=0,$ because for a random process $m_1\{\varphi'(t)\}=0,$ consequently,

$$\dot{B}'_{\mathrm{M}}(0)=\dot{B}'_{V}(0)=0.\qquad(17)$$

Differentiating (15) by τ, we get

$$\begin{aligned}&\dot{B}''_{M}(0)=\dot{B}''_{V}(0)=m_1\{\eta(t)\eta''(t)\}-\\&-2jm_1\{\eta(t)\eta'(t)\varphi'(t)\}-\\&-jm_1\{\eta(t)\eta'(t)\varphi''(t)\}-m_1\{\eta^2(t)[\varphi'(t)]^2\}.\end{aligned}$$

Since $\eta(t)$ and $\varphi(t)$ are independent, then

$$\begin{aligned}&\dot{B}''_{M}(0)=\dot{B}''_{V}(0)=m_1\{\eta(t)\eta''(t)\}-\\&-m_1\{\eta^2(t)\}m_1\{\varphi'(t)\}^2=\\&=m_1\{\eta(t)\eta''(t)\}-\overline{\eta^2}\sigma^2_{\omega},\end{aligned}\qquad(18)$$

where σ^2_{ω} is phase change derivative variance $d\varphi(t)/dt$.

It is apparent, that σ^2_{ω} represents the variance of change in the instantaneous frequency of the signal being processed under the influence of phase distortion.

The first term of the right part (18) can be represented as

$$m_1\{\eta(t)\eta''(t)\}=-\sigma^2_{\eta'}$$

where $\sigma^2_{\eta'}$ is the variance $\eta'(t)$, then

$$\dot{B}''_{M}(0)=\dot{B}''_{V}(0)=-\sigma^2_{\eta'}-\overline{\eta^2}\sigma^2_{\omega}.$$

Or, taking into account (6),

$$\dot{B}''_{M}(0)=\dot{B}''_{V}(0)=-\eta_0^2\left[\sigma^2_{\xi'}+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right],\qquad(19)$$

where $\sigma^2_{\xi'}$ is the variance $\xi'(t)$.

Taking into account the obtained values of the derivatives, the CF $\dot{B}_V(\tau)$ and $\dot{B}_M(\tau)$ with independent amplitude and phase distortions for values $\tau<<\tau_c$ can be represented as

$$\dot{B}_M(\tau)\simeq\overline{\eta^2}-0.5\tau^2\left(\sigma^2_{\eta'}+\overline{\eta^2}\sigma^2_{\omega}\right);$$

$$\dot{B}_V(\tau)\simeq\overline{\eta^2}-\left|\bar{\dot{M}}\right|^2-0{,}5\tau^2\left(\sigma^2_{\eta'}+\overline{\eta^2}\sigma^2_{\omega}\right),$$

or, when $\eta(t)=\eta_0[1+\xi(t)]$,

$$\dot{B}_M(\tau)\simeq\eta_0^2(1+\sigma^2_{\xi})-0.5\tau^2\eta_0^2\left[\sigma^2_{\xi'}+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right];\qquad(20)$$

$$\dot{B}_V(\tau)\simeq\eta_0^2(1+\sigma^2_{\xi})-\left|\bar{\dot{M}}\right|^2-0.5\tau^2\eta_0^2\left[\sigma^2_{\xi'}+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right].\qquad(21)$$

Thus, when $\tau<<\tau_c$, the function $\dot{B}_V(\tau)$ can be simply expressed in terms of the variance of the function $\xi(t)$ that determines the amplitude distortions, the variance of the derivative of the function $\xi(t)$ and the variance of the derivative of the phase deviation σ^2_{ω}.

With just phase distortions, expressions (20) and (21) are simplified as:

$$\dot{B}_M(\tau)\simeq1+0.5\tau^2\sigma^2_{\omega};$$

$$\dot{B}_V(\tau)\simeq1-\left|\bar{\dot{M}}\right|^2-0.5\tau^2\sigma^2_{\omega}.$$

If the distribution of phase and amplitude distortions is normal, (17), (19) can be obtained directly by differentiating (14) for $\dot{B}_V(\tau)$.

For the case when phase and amplitude distortions are normally distributed and correlated, the following coefficients of the expansion of $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ in the Taylor series can be obtained from (12) and (13):

$$\begin{aligned}&\dot{B}_M(0)=\eta_0^2(1+\sigma^2_{\xi});\\&\dot{B}_V(0)=\eta_0^2\{1+\sigma^2_{\xi}-[1+\sigma^2_{\xi}\sigma^2_{\varphi}r^2_{\xi\varphi}(0)]\exp\{-\sigma^2_{\varphi}\}\};\\&\dot{B}'_M(0)=\dot{B}'_V(0)=2j\eta_0^2\sigma_{\xi}\sigma_{\varphi}r_{\xi\varphi}(0);\\&\dot{B}''_M(0)=\dot{B}''_V(0)=-\eta_0^2\left[\sigma^2_{\xi'}+2\sigma^2_{\xi}\sigma^2_{\omega}r_{\xi\omega}(0)+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right],\end{aligned}$$

in the expressions presented, the coefficient $r_{\xi\omega}(0)$ of cross correlation of the function $\xi(t)$, which is determined at some value $\varphi'(t)$ (derivative of signal phase changes).

Approximate expressions for CF $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ when $\tau<<\tau_c$ will be:

$$\begin{aligned}&\dot{B}_M(\tau)\simeq\eta_0^2\{1+\sigma^2_{\xi}+2j\tau\sigma_{\xi}\sigma_{\omega}r_{\xi\omega}(0)-\\&-0.5\tau^2\left[\sigma^2_{\xi'}+2\sigma^2_{\xi}\sigma^2_{\omega}r_{\xi\omega}(0)+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right]\},\end{aligned}\qquad(22)$$

$$\begin{aligned}&\dot{B}_V(\tau)\simeq\eta_0^2\{1+\sigma^2_{\xi}-[1+\sigma^2_{\xi}\sigma^2_{\varphi}r_{\xi\varphi}(0)]\times\\&\times\exp\{-\sigma^2_{\varphi}\}+2j\tau\sigma_{\xi}\sigma_{\omega}r_{\xi\omega}(0)-\\&-0.5\tau^2\left[\sigma^2_{\xi'}+2\sigma^2_{\xi}\sigma^2_{\omega}r_{\xi\omega}(0)+(1+\sigma^2_{\xi})\sigma^2_{\omega}\right]\}.\end{aligned}\qquad(23)$$

Approximate expressions for $\dot{B}_M(\tau)$ and $\dot{B}_V(\tau)$ when $\tau<<\tau_c$ and when phase and amplitude distortions are normally distributed can also be obtained from (12) and (13) by expanding the CF of phase and amplitude distortions and their cross CF into a Taylor series.

Taking into account the fact that $B'_{\varphi}(0)=B'_{\xi}(0)=0,$ these expansions have the form:

$$
\begin{aligned}
&B_{\varphi}(\tau)=\sigma_{\varphi}^{2} r_{\varphi}(\tau)= \\
&=B_{\varphi}(\tau)+\frac{1}{2} \dot{B}_{\varphi}''(0)+\ldots=\sigma_{\varphi}^{2}-0.5 \sigma_{\omega}^{2} \tau^{2}+\ldots; \\
&B_{\xi}(\tau)=\sigma_{\xi}^{2} r_{\xi}(\tau)= \\
&=B_{\xi}(\tau)+\frac{1}{2} \dot{B}_{\xi}''(0)+\ldots=\sigma_{\xi}^{2}-0.5 \sigma_{\xi'}^{2} \tau^{2}+\ldots; \\
&B_{\xi\varphi}(\tau)=\sigma_{\xi} \sigma_{\varphi} r_{\xi\varphi}(\tau)= \\
&=B_{\xi\varphi}(\tau)+\dot{B}_{\xi\varphi}'(0) \tau+0.5 \dot{B}_{\xi\varphi}''(0) \tau^{2} \ldots= \\
&=\sigma_{\xi} \sigma_{\varphi} r_{\xi\varphi}(0)-\sigma_{\xi} \sigma_{\varphi} r_{\xi\varphi}(0) \tau-0.5 \sigma_{\xi'} \sigma_{\omega} r_{\xi'\omega}(0) \tau^{2} \ldots .
\end{aligned}
$$

Substituting these values in (12) and (13) and limiting ourselves to terms containing τ to the power not higher than the second, we obtain

$$
\begin{aligned}
&\dot{B}_{M}(\tau) \simeq \eta_{0}^{2}\left\{1+\sigma_{\xi}^{2}+2 j \tau \sigma_{\xi} \sigma_{\omega} r_{\xi\omega}(0)-\frac{1}{2} \tau^{2} \times\right. \\
&\left.\times\left[\sigma_{\xi'}^{2}+2 \sigma_{\xi}^{2} \sigma_{\omega}^{2} r_{\xi\omega}(0)\right]\right\} \exp \left\{-\frac{1}{2} \sigma_{\omega}^{2} \tau^{2}\right\};
\end{aligned} \qquad (24)
$$

$$
\begin{aligned}
&\dot{B}_{V}(\tau) \simeq \eta_{0}^{2}\left\{1+\sigma_{\xi}^{2}+2 j \tau \sigma_{\xi} \sigma_{\omega} r_{\xi\omega}(0)-\right. \\
&\left.-0.5 \tau^{2}\left[\sigma_{\xi'}^{2}+2 \sigma_{\xi}^{2} \sigma_{\omega}^{2} r_{\xi\omega}(0)\right]\right\} \exp \left\{-\frac{1}{2} \sigma_{\omega}^{2} \tau^{2}\right\}- \qquad (25) \\
&-\eta_{0}^{2}\left[1+\sigma_{\xi}^{2} \sigma_{\omega}^{2} r_{\xi\omega}(0)\right] \exp \left\{-\frac{1}{2} \sigma_{\varphi}^{2}\right\}.
\end{aligned}
$$

It is apparent that if we replace the multiplier $-0.5\sigma_{\omega}^{2}\tau^{2}$ by its expansion into a series and limit ourselves to terms containing τ to the power not higher than the second, then (24) and (25) turn into (22) and (23).

IV. Conclusions

The effect of MN when finding CF of the NMF is considered. Additionally, the definition of the CF of the named function allows to characterize the MN affecting the signal. In this article we consider the effect of only stationary and slow MN on the signal. In this case, variations in phase and distortion amplitude may be independent or related to the normal (or close to normal) distribution and are a narrow-band random process. It is shown that in order to find the CFs of NMF and fluctuations of the NMF, characteristic functions of amplitude and phase distortion can be applied, namely a four-dimensional characteristic function. At the same time, it is shown that the use of a two-dimensional characteristic function is sufficient to determine the mathematical expectation of the NMF. It is shown that the use of characteristic functions that correspond to the distribution law of phase and amplitude distortions makes it possible to determine the desired CF. The corresponding expressions are obtained for the cases when the phase and amplitude distortions can be both uncorrelated and mutually correlated. The expressions for CFs of the NMF and fluctuations of the NMF under the influence of slow MN in the case of independent variations in phase and distortion amplitude are obtained. It is shown that the CF can be expressed through the variance of the function determining the amplitude distortion, the variance of its derivative and the variance of the derivative of phase deviation. We obtained the expressions of CFs of the NMF and its fluctuations under the influence of slow MN for normal distribution law of phase and amplitude distortions and their correlation.

Determination of CF of NMF both in case of mutual connection of phase and amplitude distortion change and in case when the latter represent narrow-band random process are a new scientific result. Practical significance of the study: mathematical expressions have been obtained that allow determining the CFs of the NMF and fluctuations of the NMF for sufficiently important practical applications.

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