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Mathematical Methods for Describing the non-Gaussian Random Variables and Processes

Vladimir Mikhaylovich Artyushenko
Information technology and management systems department
Technological University
Korolev city, Russian Federation
artuschenko@mail.ru

Vladimir Ivanovich Volovach
Informational and electronic service department
Volga Region State University of Service
Togliatty city, Russian Federation
volovach.vi@mail.ru

Kseniya Vladimirovna Lyapina
Informational and electronic service department
Volga Region State University of Service
Togliatty city, Russian Federation
kvsafronova@yandex.ru

Alyona Igorevna Kutukova
Informational and electronic service department
Volga Region State University of Service
Togliatty city, Russian Federation
alyona_matveeva_95@mail.ru

Abstract—**Mathematical methods for describing univariate and bivariate non-Gaussian random variables and processes in their modeling are considered. A decomposition of probability density functions (PDFs) using orthogonal polynomials is described for both univariate and bivariate PDFs. It is noted that, in both cases, decompositions using the specified orthogonal functions can be used in most cases. The area of application of the exponential PDF is shown. We give a representation of two-dimensional PDF in the form of an Edgeworth-type expansion by Hermite polynomials. The correlation between the correlation function and the bivariate PDF of a random process is shown. We analyze the representation of one-dimensional distributions by orthogonal Gramm-Charlier and Edgeworth series as well as of two-dimensional distributions in the form of a Hermite polynomial expansion. It is noted that Edgeworth's series provides a better approximation of PDFs than the Gramm-Charlier series. It is shown that the coefficient of excess characterizes the shape of PDF. We consider the method of PDF decomposition by Laguerre polynomials which are used only for one-way PDFs. The field of Fourier series decomposition is determined. The formation and superposition methods used in the formation of random variables are described.**

*Keywords—**probability density function, mathematical modeling, random variable, orthogonal polynomial, Hermite polynomials, Edgeworth series, Gramm-Charlier series, Laguerre polynomials, kurtosis factor, Fourier series, superposition method***

I. Introduction

Multivariate probability density functions (PDFs) are known to be widely used to describe non-Gaussian random processes [1-4 etc.]. However, it is difficult or impossible to define multidimensional PDFs in real-world applications. Therefore, various approximate methods are used to describe them, which are currently used in radiophysics, radio engineering, radiolocation, telecommunications and information and measurement systems [5-10 etc.]. This paper will discuss some approximate methods for describing non-Gaussian random variables and processes.

Many scientific publications are devoted to methods of formation and modeling of random variables (RV) and processes with a given PDF [11-18 etc.]. Independent standard random variables with uniform distribution on the interval [0, 1] are used to obtain realizations of random variables

$$\gamma \sim \mathrm{Rav}\left[a, b\right], \quad a = 0,\ b = 1,$$

where Rav is the mathematical sign of a uniform distribution of a random variable on a given interval,
and Gaussian independent RV

$$\xi \sim N\left(m, \sigma^2\right),$$

where m, σ^2 is, respectively, mathematical expectation and variance.

Non-linear transformation, superposition, piecewise approximation, Neumann and other methods are used to simulate random variables.

Let's look at and analyze the most common methods.

The purpose of this paper is to review the mathematical methods used to describe univariate and bivariate non-Gaussian random variables and processes in modeling the latter, and to determine the applicability of individual methods.

II. Decomposition of Probability Density Distributions by Means of Orthogonal Polynoms

Apply the expression

$$W(\xi) = W_r(\xi) \sum_{k=0}^{\infty} c_k Q_k(\xi),$$

where $\{Q_k(\xi)\}$ is a system chosen in some way of orthonormalised, with weight $W_r(\xi)$ polynomials, such that

$$Q_k(\xi) = \sum_{i=0}^{k} a_i \xi^i$$

for decomposition according to some «reference» law $W_r(\xi)$ to a given one-dimensional PDF $W(\xi)$.

Use the expression below to calculate the coefficient c_k

$$c_k = \int_{-\infty}^{\infty} W(\xi) Q_k(\xi)\, d\xi.$$

In addition, the coefficients c_k can be found from the expression

$$c_k = \sum_{i=0}^{k} a_i m_{iw},$$

if the following expression is used to find the i-th initial moment $W(\xi)$ of the RDF

$$\int \xi^i W(\xi)\, d\xi = m_{iw}.$$

From

$$W(\xi) = W_r \sum_{k=0}^{\infty} Q_k(\xi) \sum_{i=0}^{k} a_i m_{iw}.$$

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It is important to note that decomposition

$$W(\xi) = W_r(\xi) \sum_{k=0}^{N} c_k Q_k(\xi)$$

on N summands fully satisfies the normalization condition, which is written as follows

$$\int_{-\infty}^{\infty} W(\xi) d\xi = 1.$$

By analogy, it becomes possible to write decompositions that are also feasible for two-dimensional PDFs:

$$W(\xi_1, \xi_2) = W_r(\xi_1) W_r(\xi_2) \sum_{k,r=0}^{\infty} c_{kr} Q_{1.k}(\xi_1) Q_{2.k}(\xi_2).$$

In the latter expression, the authors use one-dimensional PDFs $W(\xi_1)$ and $W(\xi_2)$ as «reference» PDFs $W_r(\xi_1)$ and $W_r(\xi_2)$ according to earlier studies [10]. In turn, the named one-dimensional PDFs should correspond to a specific two-dimensional PDF $W(\xi_1, \xi_2)$. A selection condition the polynomials $Q_{1.k}(\xi_1)$ and $Q_{2.k}(\xi_2)$ is that the coefficients are equal to zero $c_{kr} = 0$ at a given $k \neq r$.

In this case we get the expression

$$W(\xi_1, \xi_2) = W(\xi_1) W(\xi_2) c_n Q_{1.n}(\xi_1) Q_{2.n}(\xi_2), \quad (1)$$

where $c_n = \iint_{-\infty}^{\infty} W(\xi_1, \xi_2) Q_{1.n}(\xi_1) Q_{2.n}(\xi_2) d\xi_1 d\xi_2$.

It should be noted that the same kind of expansion as above can also be written for characteristic functions.

In addition, the above expressions for the decomposition of two-dimensional PDF can sometimes also be performed for the case of non-orthogonal polynomials $Q_{1.k}(\xi_1)$ and $Q_{2.k}(\xi_2)$. However, this case will not be considered by the authors in this paper.

Note that ambiguity in the choice of an approximating family arises in the absence of complete a priori information about one-dimensional PDFs. An example of this ambiguity is the description of the a posteriori PDF used in signal demodulation. An example of this ambiguity is the description of the a posteriori PDF used in signal demodulation.

In [19] we show the use of an exponential PDF to approximate an unknown PDF $W(\xi|\alpha)$ with some uncertain parameter α and then determine the unknown parameters of the exponential PDF.

For example, the unknown PDF can be written as is represented as

$$W(\xi|\alpha) = \exp\left\{ \sum_{i=1}^{N} \varphi(\xi)\alpha_i + \Gamma(\alpha) + \varphi_c(\xi) \right\},$$

where $\Gamma(.)$ is Gamma function.

It is important to note that in this expression, the following systems of functions must be linearly independent $(1, \varphi_1, \ldots, \varphi_N)$, $(\alpha_1, \ldots, \alpha_N)$.

Using the normalization condition, it is relatively easy to find the gamma function $\Gamma(\alpha)$:

$$\Gamma(\alpha) = -\ln \int_{-\infty}^{\infty} \exp\left\{ \sum_{i=1}^{N} \alpha_i \varphi_i(\xi) + \varphi_0(\xi) \right\} d\xi = 1.$$

The dependence of the process at adjacent points in time is reflected in the use of a function $\varphi(\xi_h, \xi_{h-1})$. The latter is applied if there is a sampling dependence in the exponent for the PDF $W(\xi|\alpha)$ sought.

We will assume, as before, that the type of function $\varphi(\xi_h, \xi_{h-1})$ remains a priori unknown. Then it is possible to decompose the said function into basis functions using uncertain coefficients $\{\beta_{ij}\}$, $i, j = \overline{0, N}$ (the line at the top represents an averaging over the set):

$$\psi(\xi_h, \xi_{h-1}) = \sum_{i,j=1}^{N} \beta_{ij} \varphi_i(\xi_h) \varphi_\alpha(\xi_{h-1}).$$

In the case under consideration, the PDF $W(\xi|\alpha)$ is converted as follows [10]:

$$W(\xi_h, \xi_{h-1}) = \exp\Big\{ \Gamma(\alpha, \beta, \xi) + \sum_{i=1}^{N} \alpha_i \varphi_i(\xi_h) + \\ + \sum_{i=1}^{N} \sum_{j=1}^{N} \beta_{ij} \varphi_i(\xi_h) \varphi_j(\xi_{h-1}) \Big\}.$$

The expression includes uncertain parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_{ij})$ and a given family of linearly independent functions $\varphi_j(x)$, $j = \overline{1, N}$.

Again, as in the reasoning above, the normalization condition helps to find the gamma function $\Gamma(\alpha, \beta)$:

$$\int_{-\infty}^{\infty} W(\xi_h | \xi_{h-1}) = d\xi_h = 1.$$

Using [4], it can be argued that the exponential family belongs to the Pearson PDF (in which the Pearson criterion is equal to $z = \pm\infty$):

$$W(\xi) = W_0 \left(1 + \frac{\xi}{l}\right)^q \exp\left\{ -\frac{q\xi}{l} \right\},$$

where

$$\xi = \xi' - m_1 + \frac{dk_a}{2}; \quad W_0 = \frac{1}{l} \times \frac{q+1}{\exp\{q\} \Gamma(q+1)};$$

$$l = \sigma\left(\frac{2}{k_a} - \frac{k_a}{2} \right); \quad q = \frac{4}{k_\alpha^2} - 1; \quad k_a = \frac{M_3}{\sigma^3},$$

k_a is asymmetry coefficient, M_3 is 3rd selective central moment, σ is standard deviation, m_1 is sample value of the first starting torque, (′) is mathematical sign of the time derivative.

It should be clarified that the Pearson PDF used is n-dimensional normal, exponential, multinomial and bimodal.

We use an Edgeworth-type expansion of Hermite polynomials to describe the two-dimensional PDF $W(t, \xi_1, \xi_2)$:

$$W(t, \xi_1, \xi_2) = \frac{1}{\sqrt{2\pi \varkappa_{20} \varkappa_{02}}} \exp\left\{ -\frac{1}{2} \times \frac{(\xi_1 - \varkappa_{10})^2}{\varkappa_{20}} + \frac{(\xi_1 - \varkappa_{01})^2}{\varkappa_{02}} \right\} \left\{ 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \rho_{11}^k H_k\left(\frac{\xi_1 - \varkappa_{10}}{\sqrt{\varkappa_{20}}} \right) H_k\left(\frac{\xi_1 - \varkappa_{02}}{\sqrt{\varkappa_{02}}} \right) + \right.$$

$$+ \sum_{m_1+m_2=3} \frac{1}{m_1! m_2!} \frac{\varkappa_{m_1 m_2}}{\sqrt{\varkappa_{20}^{m_1} \varkappa_{02}^{m_2}}} \sum_{k=0} \frac{1}{k!} \rho_{11}^k H_{m_1+k}\left(\frac{\xi_1 - \varkappa_{10}}{\sqrt{\varkappa_{20}}} \right) H_{m_2+k}\left(\frac{\xi_2 - \varkappa_{01}}{\sqrt{\varkappa_{02}}} \right) + \sum_{m_1+m_2=4} \frac{1}{m_1! m_2!} \frac{\varkappa_{m_1 m_2}}{\sqrt{\varkappa_{20}^{m_1} \varkappa_{02}^{m_2}}} \sum_{k=0} \frac{1}{k!} \rho_{11}^k H_{m_1+k}\left(\frac{\xi_1 - \varkappa_{10}}{\sqrt{\varkappa_{20}}} \right) \times$$

$$\times H_{m_2+k}\left(\frac{\xi_2-\varkappa_{01}}{\sqrt{\varkappa_{02}}}\right)+\frac{1}{2}\sum_{m_1+m_2=3}\sum_{n_1+n_2=3}\frac{\varkappa_{m_1m_2}\varkappa_{n_1n_2}}{\left\{\varkappa_{20}^{m_1}\varkappa_{02}^{m_2}\varkappa_{20}^{n_1}\varkappa_{02}^{n_2}m_1!m_2!n_1!n_2!\right\}^{0.5}}\sum_{k=0}\frac{1}{k!}\rho_{11}^{k}H_{m_1+n_1+1}\left(\frac{\xi_1-\varkappa_{10}}{\sqrt{\varkappa_{20}}}\right)H_{m_2+n_2+1}\left(\frac{\xi_2-\varkappa_{01}}{\sqrt{\varkappa_{02}}}\right)+\ldots\right\},$$

$$\rho_{11}=\frac{\varkappa_{11}}{\sqrt{\varkappa_{20}\varkappa_{02}}}.$$

Here the symbols where $\sum_{m_1+m_2=M}$; $\sum_{n_1+n_2=N}$, where $M = 3, 4$; $N = 3$, mean that the summation is performed over all such values m_1, m_2, n_1, n_2 that $m_1+m_2=M$; $n_1+n_2=N$, $H_n(z)$ are one-dimensional Hermite polynomials.

Considering that

$$\varkappa_{11}=\langle\xi_1(t_1)\xi_2(t_2)\rangle B_{12}(\tau)=R_{12}(\tau)+m_1m_2$$

is the covariance matrix, which at

$$m_1=m_2=0$$

coincides with the correlation function $B_{12}(\tau)$

$$\varkappa_{02}=\sigma_{\xi_2};\varkappa_{20}=\sigma_{\xi_1},$$

and ρ_{11} coincides with the coefficient $R_{12}(\tau)$ (so called the dimensionless correlation coefficient), and considering that $H_0(\xi) = 1$, $H_1(\xi) = \xi$ for the case where [10]

$$\varkappa_{01}=\varkappa_{10}=m_1=m_2=0,$$

let`s find

$$W_\xi(\xi_1,\xi_2)=\frac{1}{2\pi}\exp\left\{-\frac{\xi_1^2}{2}-\frac{\xi_2^2}{2}\right\}\{1+B_\xi(\tau)\xi_1\xi_2+ +2\pi\sum_{n=2}^{\infty}c_nH_n(\xi_1)H_n(\xi_2)\ldots\},\tag{2}$$

where $\langle.\rangle$ is a mathematical sign indicating averaging over a set.

Relationship (2) reflects the relationship between the correlation function $B_\xi(\tau)$ and the bivariate PDF of the random process $W_\xi(\xi_1, \xi_2)$.

Expression (2) makes it possible to determine the difference between the two-dimensional PDF of the non-Gaussian process [20] under study and its model. In this case the model of the named process should be built using the criterion of coincidence of one-dimensional PDF of the real process and its correlation function.

Obviously, the error

$$\Delta(\xi_1,\xi_2)=2\pi\sum c_nH_n(\xi_1)H_n(\xi_2)$$

depends on the coupling moment of the form $m\left[\xi_1^k\xi_2^r\right]$, where $k\neq r\neq 1,\ k,r=1,2,\ldots,\infty$.

III. Representation of one-dimensional probability densities by orthogonal Netherlands

With little difference from Gaussian PDFs, the approximation of PDFs by series of orthogonal polynomials can be used to describe unimodal PDFs. The coefficients of the latter will be determined by the moments of the distributions.

1. Gramm-Charlier and Edgeworth series. Let us write the Gramm-Charlier series using the family of orthogonal polynomials of the Gaussian PDF as the weight function:

$$W(\xi)=\frac{1}{\sqrt{2\pi}}\exp\{-0.5\xi^2\}\sum_{i=0}^{\infty}c_iH_i(\xi),$$

where $c_n=\frac{1}{n!}\int_{-\infty}^{\infty}W(\xi)H_n(\xi)d\xi$ is the decomposition factor,

$H_n(y)=(-1)^n\exp\{-y^2\}\frac{d^n}{dy^n}\exp\{-y^2\}$ is the Hermite polynomial satisfying the recurrence relation

$$H_{n+1}(y)=2yH_n(y)-2nH_{n-1}(y),\ n\geq 1;$$
$$H_0(y)=1;\quad H_1(y)=y;\quad H_2(y)=y^2-1;$$
$$H_3(y)=y^3-1;\quad H_4(y)=y^4-5y^2+3.$$

The use of cumulants $\varkappa_{\xi k}$ of PDF $W(\xi)$ allows decomposition coefficients to be obtained [21].

A normalized random variable can be written in the following form:

$$c_{01}=1;\quad c_1=c_2=1;\quad c_3=\frac{\varkappa_{\xi3}}{3!};\quad c_4=\frac{\varkappa_{\xi4}}{4!};$$
$$c_5=\frac{\varkappa_{\xi5}}{5!};\quad c_6=\frac{(\varkappa_{\xi6}+10\varkappa_{\xi3}^2)}{6!};\quad c_7=\frac{(\varkappa_{\xi7}+35\varkappa_{\xi4}\varkappa_{\xi3})}{7!};$$
$$c_8=\frac{(\varkappa_{\xi8}+56\varkappa_{\xi5}\varkappa_{\xi3}+35\varkappa_{\xi4}^2)}{8!};$$
$$c_9=\frac{(\varkappa_{\xi9}+84\varkappa_{\xi6}\varkappa_{\xi3}+126\varkappa_{\xi5}\varkappa_{\xi4}+280\varkappa_{\xi3}^3)}{9!}.$$

For the case of a normalized random variable, let's express the PDF $W(\xi)$ as a series of Edgeworth PDFs

$$W(\xi)=\frac{1}{\sqrt{2\pi}}\exp\{-0.5\xi^2\}\times$$
$$\times\left\{1+\frac{\varkappa_3}{3!}H_3(\xi)+\frac{\varkappa_4}{4!}H_4(\xi)+\frac{10\varkappa_3^2}{6!}H_6(\xi)+\right.$$
$$\left.+\frac{\varkappa_5}{5!}H_5(\xi)+\frac{35\varkappa_4\varkappa_5}{7!}H_7(\xi)+\frac{\varkappa_6}{6!}H_6(\xi)+\ldots\right\}.$$

In the absence of normalization the Edgeworth series can be written as follows

$$W(\xi)=N(m,\sigma^2)\left[1+\sum_{n=3}^{N}\frac{1}{3!}\frac{b_n}{\sigma^n}H_n\left(\frac{\xi-m}{\sigma}\right)\right];$$

$$N(m,\sigma^2)=\frac{\exp\left\{-\frac{(\xi-m)^2}{2\sigma^2}\right\}}{\sigma\sqrt{2\pi}},$$

where $b_n = \sigma^n \int_{-\infty}^{\infty} N(m,\sigma^2) H_n\left(\frac{\xi - m}{\sigma}\right) d\xi$ is decomposition coefficients, also called quasimoments.

If a Gaussian distribution is considered, all quasimoments are zero when the inequality $n \geq 3$.

Let's write down the first two coefficients, called the coefficient of asymmetry k_a (which we have already mentioned earlier) and the coefficient of kurtosis k_k:

$$k_a = \frac{b_3}{\sigma^3} = \frac{M_3}{\sigma^3} = \frac{\varkappa_3}{\varkappa_2^{3/2}}; \quad k_k = \frac{b_4}{\sigma^4} = \frac{M_4}{\sigma^4} - 3 = \frac{\varkappa_4}{\varkappa_2^2}.$$

The coefficient of kurtosis characterizes the shape of the PDF. For example, if $k_k > 0$, then the PDF is characterized by a «pointed» top. Otherwise (at $k_k < 0$) the top of the PDF will be flatter.

To determine the i-th order central momentum, use the expression $M_i = \int_{-\infty}^{\infty} (\xi - m)^i W(\xi) d\xi$.

The relationship between the cumulants $\varkappa_{\xi i}$ of a normalized random variable ξ and the cumulants $\varkappa_{\tilde{\xi} i}$ of a non-normalized random variable ξ is defined by the relation

$$\varkappa_{\xi i} = \frac{\varkappa_{\tilde{\xi} i}}{\sqrt{\varkappa_{\tilde{\xi}2}}^{2}}.$$

Here is an expression linking the above expressions of the PDF of the non-normalized $W(\xi)$ and normalized $W(\xi)$ random variables

$$W(\xi) = \frac{1}{\sqrt{\varkappa_{\tilde{\xi}2}}} W\left(\frac{\xi - \varkappa_{\tilde{\xi} i}}{\sqrt{\varkappa_{\tilde{\xi}2}}}\right)$$

If a fixed maximum order of applied cumulants is used, the Edgeworth series gives a better approximation of the PDF $W(\xi)$ than the Gramm-Charlier series.

2. Laguerre polynomial expansion. It was noted earlier by the authors [10] that the Edgeworth series has slow convergence for one-sided and defined only for positive values of the PDF argument. Using Laguerre polynomials to decompose the PDF in this case would be preferable.

The PDF of a gamma distribution is a weight function of a family of orthogonal polynomials:

$$W(\xi) = \xi^\alpha \exp\{-\xi\} \sum_{i=0}^{\infty} c_i L_i^{(\alpha)}(\xi); \quad \xi \geq 0,$$

which uses a generalized Laguerre polynomial

$$L_n^{(\alpha)} = \exp\{\xi\} \frac{\exp\{-\alpha\}}{n!} \frac{d^n}{d\xi^n}\left(\exp\{-\xi\} \xi^{n+\alpha}\right); \quad \alpha > -1.$$

Write down the first five Laguerre polynomials

$$L_c^{(\alpha)}(\xi) = 1; \quad L_1^{(\alpha)}(\xi) = (a+1) - \xi;$$

$$L_2^{(\alpha)}(\xi) = \frac{1}{2}\left[(a+2;-1;2) - 2(a+2)\xi + \xi^2\right];$$

$$L_3^{(\alpha)}(\xi) =$$
$$= \frac{1}{6}\left[(a+3;-1;3) - 3(a+3;-1;2)\xi + 3(a+3)\xi^2 - \xi^3\right];$$

$$L_4^{(\alpha)}(\xi) = \frac{1}{24}\left[(a+4;-1;4) - 4(a+4;-1;3)\xi + \right.$$
$$\left. +6(a+4;-1;2)\xi^2 - 4(a+4)\xi^3 + \xi^4\right];$$

$$L_5^{(\alpha)}(\xi) = \frac{1}{120}\left[(a+5;-1;5) - 5(a+5;-1;4)\xi + \right.$$
$$\left. +10(a+5;-1;3)\xi^2 - 10(a+5;-1;2)\xi^3 + 5(a+5)\xi^4 - \xi^5\right],$$

where the designation

$$(a;b;n) = a(a+b)(2a+b)\ldots\left[a+(n-1)b\right].$$

Decomposition coefficients

$$c_n = \frac{n!}{\Gamma(n+\alpha+1)} \int_0^\infty L_n^\alpha(\xi) W(\xi) d\xi.$$

Here is an expression to approximate the simplest approximation of the PDF $W(\xi)$ by the Laguerre polynomial:

$$W(\xi) = \frac{1}{\beta\Gamma(\alpha+1)} \left(\frac{\xi}{\beta}\right)^\alpha \exp\left\{-\frac{\xi}{\beta}\right\}. \quad (3)$$

In formula (3), the parameters α and β are expressed in terms of the mathematical expectation m_1 and variance σ^2 of the PDF $W(\xi)$

$$\alpha = \frac{m_1^2}{m_2 - m_1} - 1 = \frac{m_1^2}{\sigma^2} - 1; \quad \beta = \frac{m_2 - m_1^2}{m_1} = \frac{\sigma^2}{m_1}.$$

The known functions $\varepsilon_k(\xi, t)$ and unknown coefficients $\hat{g}(\lambda, t)$ allow one-dimensional PDFs $W(\xi, t)$ to be written as follows:

$$W(\xi, t) = \sum_{k=1}^{M} g(\lambda_k, t) \varepsilon_k(\xi, t),$$

where

$$\varepsilon_k(\xi, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\{-i\lambda_k\} \psi_k(\lambda, t) d\lambda.$$

$g(\lambda, t)$ refers to the one-dimensional characteristic function of the PDF $W(\xi, t)$. The named characteristic function, in turn, can be written as an equivalent sum:

$$g(\lambda, t) = \sum_{k=1}^{M} g(\lambda_k, t) \psi_k(\lambda, t).$$

Here are some fixed values of the argument of the characteristic function $\lambda_1, \ldots, \lambda_m$, as well as known functions $\psi_k(\lambda, t)$ that obey the conditions $\psi_k(\lambda_v, t) = \delta_{kv}$ (δ_{kv} is the Kronecker symbol).

As a result, determining the values of one-dimensional characteristic functions at certain points $\lambda_1, \ldots, \lambda_m$ allows us to describe the representation of the PDF at given functions $\psi_k(\lambda, t)$.

3. Fourier series expansion. We use the Fourier series to represent a one-dimensional PDF that is different from zero at the interval [a, b]:

$$W(\xi) = \sum_{k=-\infty}^{\infty} a_k \exp\left\{-\frac{2\pi jk}{h}\xi\right\}; \quad h = b - a, \quad \xi \in [a, b],$$

$$a_k = \frac{1}{h} \int_a^b W(\xi) \exp\left\{\frac{2\pi jk}{h}\xi\right\} d\xi.$$

In addition, if it is specified that $a=-b$ and one-dimensional PDF $W(\xi)$ is an even function, then the above expressions can be simplified:

$$W(\xi)=a_0+2\sum_{k=1}^{\infty}a_k\cos\frac{2\pi k\xi}{h},$$

$$a_k=\frac{2}{h}\int_0^b W(\xi)\cos\frac{2\pi k\xi}{h}d\xi.$$

When calculating an estimate of the PDF, their estimates a_k expressed as polynomial or semi-invariant estimates, are used instead of a_k coefficients.

IV. Methods of Tranformation and Superposition in the Formation of Random Variables

It is based on the determination of the PDF RV $\xi = \psi(\eta)$ derived from the non-linear transformation of RV η.

A transform $\psi(x)$ is said to be smooth, monotonically increasing $\psi'(x)>0$, where (′) is the mathematical sign of the time derivative, if it has an inverse transform such that $x=\psi'(y)$.

Since for the RV distribution functions ξ and η the equality is valid

$$F_\xi(x)=F_\eta\left[\psi^{-1}(x)\right], \tag{4}$$

then by differentiating, we get

$$W_\xi(x)=W_\eta\left[\psi^{-1}(x)\right]\frac{d\psi^{-1}(x)}{dx}.$$

By putting in expression (1)

$$\psi(x)=F_\eta(x),$$

get $F_\xi(x)=x.$

That is, the random variable ξ has a uniform distribution.

As a result, if a random variable η is transformed by a non-linear transformation equal to its distribution function, we obtain on the interval [0, 1] a uniformly distributed RV.

If in expression (4) you put

$$\psi(x)=F^{-1}(x)$$

and the given distribution function $F(x)$ is used here, and

$$\eta\in\mathrm{Rav}[0,1],\quad F_\eta(x)=x,$$

we obtain that the RV ξ we obtain that the

$$F_\xi(x)=\left[F^{-1}(x)\right]^{-1}=F(x).$$

This rule is the basis of the method of forming a RV with a given distribution function.

Note that in this case it is necessary to perform a non-linear transformation $\xi=F^{-1}(\gamma)$, meaning the solution of the equation in the form

$$F(\xi)=\gamma,\quad \gamma\in\mathrm{Rav}[0,1].$$

Let us consider an example. Suppose we want to generate a random variable with a PDF

$$W(x)=\begin{cases}\dfrac{x}{a\sqrt{a^2-x^2}}, & x\in[0,a);\\ 0, & x[0,a).\end{cases}$$

By integrating $W(x)$, we obtain an expression for the distribution function

$$F(x)=\frac{1}{a}\int_0^x\frac{t}{\sqrt{a^2-t^2}}dt=1-\frac{1}{a}\sqrt{a^2-x^2}.$$

From this we get the equation

$$1-\sqrt{\frac{a^2-x^2}{a^2}}=\gamma_1,\quad \gamma_1\in\mathrm{Rav}[0,1],$$

from which the modeling algorithm follows

$$\xi=a\sqrt{1-\gamma^2},$$

where $\gamma=1-\gamma_1$.

The superposition method, which is used to generate random variables with a PDF of the following form, is widespread

$$W(\xi)=\sum_{i=1}^{n}p_iW_i(\xi),\quad p_i>0,\ \sum_{i=1}^{n}p_i=1. \tag{5}$$

It is performed in two steps.

In the first step, a discrete RV is implemented, taking values $\overline{1,n}$ with probabilities.

In the second step, after obtaining the value of k, a value is generated with $W_k(\xi)$, he value of which is taken as ξ.

Models (5) are called PDF mixtures $W_1(\xi)$, …, $W_n(\xi)$.

Consider an example of the formation of a random variable that has a PDF

$$W(\xi)=\frac{0.5}{\sigma\sqrt{2\pi}}\left[\exp\left\{-\frac{(\xi+m)^2}{2\sigma^2}\right\}+\exp\left\{-\frac{(\xi-m)^2}{2\sigma^2}\right\}\right],$$

where σ is standard deviation of a random variable.

As can be seen, $W(\xi)$ is a mixture of two Gaussian PDFs with equal variances σ^2, means $\pm m$ and weights $p_1=p_2=0.5$.

According to superposition methods, the algorithm for generating a random variable is as follows

$$\xi=\sigma_x+\gamma_m,$$

where $x\in N(0,1)$, and γ_m takes equal probability two values $\pm m$ where $x\in N(0,1)$, and γ_m takes equal probability two values $\pm m$.

V. Conclusion

The paper considers issues related to mathematical methods of describing non-Gaussian random variables and processes. The use of orthogonal polynomials for decomposition of univariate PDFs is described. It has been shown that ambiguity in the selection of the approximating family appears when there is no complete a priori information about one-dimensional PDFs.

Use of Edgeworth series on Hermite polynomials for description of two-dimensional PDF is given. It is noted that the use of the matching criterion of the one-dimensional PDF of the real random process and the correlation function allows to determine with sufficient accuracy the degree of difference between the two-dimensional PDF of the named process and the constructed model.

The use of orthogonal series to represent one-dimensional PDFs is described. It has been shown that the approximation of PDF with rows of orthogonal polynomials is successfully used to describe unimodal PDF close in form to Gaussian. At the same time, the coefficients of polynomials are determined by the moments of distributions, otherwise by the Edgeworth and Gramm-Charlier series. With a fixed maximum order of cumulants used, the use of the Edgeworth series is desirable.

The use of Laguerre polynomials and Fourier series decomposition is given. It has been shown that for the decomposition of PDF according to the Laguerre polynomial, it is more acceptable for one-sided PDF defined for positive values of the argument. It is also possible to use a Fourier series to represent a one-dimensional PDF given at a certain final value interval.

A method for transforming random variables, based on determining the PDF of a RV derived from a non-linear transformation of the random variable itself, is described. The content of the superposition method, also used to form random variables, is given.

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